Engineering Deutsch-Jozsa Algorithm in Cavity QED via Bragg Regime

Final Year Project Report

Presented

by

Abdullah Naeem Malik
C.I.I.T./SP08-BSM-001/ISB

In Partial Fulfillment
of the Requirement for the Degree of
Bachelor of Science in Mathematics

DEPARTMENT OF MATHEMATICS

COMSATS INSTITUTE OF INFORMATION TECHNOLOGY,
ISLAMABAD
JANUARY 2012

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Declaration

I hereby declare that this project, neither as a whole nor as a part thereof, has been copied out from any source. It is further declared that I have developed this project and the accompanied report entirely on the basis of my personal efforts made under the sincere guidance of my supervisor. No portion of the work presented in this report has been submitted in the support of any other degree or qualification of this or any other university or institute of learning. If found guilty of breaching the aforementioned, I shall be liable for any judgement deemed fit by the competent authority against such misconduct.

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Abdullah Naeem Malik

COMSATS INSTITUTE OF INFORMATION TECHNOLOGY, ISLAMABAD
JANUARY 2012

Engineering Deutsch-Jozsa Algorithm in Cavity QED via Bragg Regime

An Undergraduate Final Year Project Report submitted to the

DEPARTMENT OF MATHEMATICS

As a partial Fulfillment for the award of Degree
Bachelor of Science in Mathematics

By

Name	**Registration Number**
Abdullah Naeem Malik	C.I.I.T/SP08-BSM-001/ISB

Supervised by

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COMSATS INSTITUTE OF INFORMATION TECHNOLOGY, ISLAMABAD
JANUARY 2012

Final Approval

This project titled

Engineering Deutsch-Josza Algorithm in Cavity QED via Bragg Regime

Submitted for the Degree of
Bachelor of Science in MATHEMATICS

By

Name	**Registration Number**
Abdullah Naeem Malik	**C.I.I.T./SP08-BSM-001/ISB**

has been approved for

COMSATS INSTITUTE OF INFORMATION TECHNOLOGY,
ISLAMABAD

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Dr. Moiz ud Din Khan
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To

Kurt Friedrich Gödel

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Abdullah Naeem Malik

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> And the strangeness will all prove to be connected and make sense
> - Edward O. Wilson

1 Introduction

Classical mechanics, a renowned branch, deals with the interaction of massive objects - from apples and balls to ships or even planets. This branch was enriched with many theories by many people, amongst which Sir Isaac Newton, PRS, was a pioneer[1]. Newton's method covered a wide range of applications to particles. Joseph-Louis Lagrange later offered a very comprehensive treatise on mechanics, comparable to Newton's. In his *Mécanique Analytique,* Lagrange developed his own method, involving, what is now-called, the Lagrangian[2]. This was later modified by Sir William Rowan Hamilton to cover the description of a system with an enormous number of particles[3]. The trajectory of any body could be determined. Even a roulette could exactly predict where the ball might have landed, given the exact position, momentum and energy of the object. It seemed as though the apparent world had been daunted - determinism had taken shape. A mathematical model of gravity on earth that could entail successfully accurate predictions for the heavenly bodies was mysterious and filled with awe, to say the least. Mathematics was "unreasonably effective"[4], to mark the words of a Hungarian American theoretical physicist and mathematician Eugene Paul Wigner, FRS, himself. The method "spread the light of mathematics on a science which up to then had remained in the darkness of conjectures and hypotheses"[5].

Now that nearly all the macroscopic phenomena could be explained using the physics passed down by scientists for 200 years upto the nineteenth century, scientists turned to light. Greek, Arab, Babylonian and Egyptian philosophers had a wide range of theories for light, amongst which Plato's theory was most popular[6]. There were many competing theories of light and all were partially or completely incorrect. The first question that demanded scientific attention was the speed of light. Initially, light was conceived of having a speed that was not finite - an uncomfortable idea today. This was conceived from an experiment by Galileo Galilee. Galileo tried to determine the speed of light by arranging two observers standing at a distance apart, holding laterns equipped with shutters, observing each other's lanterns by signaling each other. He was unable to measure any significant time lapse of the signals, compared to the human reflexes[7]. After all, even a quick movement of the head in the mirror will convince us of the same! Later in time, however, the finiteness of the speed of light was proposed by Ole Roemer who observed a moon of Jupiter, Io, in 1675[8]. Impressively, his calculations yielded a speed of $2 \times 10^8 m/s$[8]. There is evidence that Newton favoured this calculation:

> "Light is propagated from luminous bodies in time and spends about seven or eight minutes of an hour in passing from the Sun to the Earth. This was observed first by Romer, and then by others, by means of the Eclipses of the Satellites of Jupiter"[9].

The interaction of matter with matter became clear with Newton's *Principia Mathematica*. However, the then understanding of the nature of light and consequently its interaction with matter was not without its difficulties. Consequently, Newton turned to the nature of light - a domain that remained disputed, notwithstanding his presence. There is even evidence that he did not publish his *Opticks* during the life of Robert Hooke, FRS, for fear of criticism[10]. It should be remarked that Hooke was a well-reputed natural philosopher, writer and polymath. In his *Opticks*, Newton theorised that light was also composed of particles, which he called corpuscles of light[9]. This model then adopted fairly with his theorems in *Principia*. To Newton, light travelled in extremely straight lines in any constant medium, because there was no net force to act on it, deep within that medium. A particle of light could be seen travelling in a projectile in the form of a parabola - the straightness was due to the fact that in one nanosecond, light travels a distance of 30 cm. The curvature would then naturally be extremely small to be detectable. The reflection of light could be paralleled to a ball bouncing off a surface. Refraction could be explained by considering different forces acting on a light corpuscle in a change of a medium. To account for a prism's splitting, Newton simply stated that the amount of colour varied with the mass of a corpuscle and hence was the cause of a different amount of "bending" (refraction). To explain polarisation, it was hypothesised that the corpuscles were not spherical but, in fact, flat, plate-like.

As beautiful as the corpuscular hypothesis may seem, coupled by Newton's reputation, this model, however, failed because it could not account for interference, diffraction and birefringence of light. In order to account for interference, diffraction and birefringence, a whole new approach to light - a wave theory - was proposed by René Descartes, the French famous for his philosophy, in 1637[11]. Basing his arguments on what is generally believed[12] to be Willebrord Snellius' manuscript of an experiment Snell conducted in 1621, the law of refraction was published by Descartes[11], thus hailing a wave nature of light. Furthermore, Hooke in 1665[13] and Robert Boyle, FRS, in 1664[14] both independently discovered what is now called "Newton's rings". Christiaan Huygens, FRS, is another figure who contributed significantly to the wave front theory of light in1678[15]. Notwithstanding the development of the wave theory of light, the arguments in its favour were crushed by Sir Isaac Newton majorly because of his reputation. The particle theory of light remained dominant for over a century. However, during this period, the wave theory of light did find a support in Leonard Euler in 1746[16]. Later when the particle theory of light was found inadequate, the wave front theory of Christiaan Huygens was adopted. This happened because of Thomas Young in 1801[17][18], who is noted for greatly challenging the corpuscular hypothesis by his principle of interference. The treatment of the French engineer Augustin-Jean Fresnel on the wave nature of light in 1850[19] received the Paris Academy 1818 prize[12]. The wave theory was also defended rigorously by James Clerk Maxwell, FRS, FRSE, in 1895[20]. In 1888, Heinrich Rudolf Hertz experimentally verified Maxwellian view of light equations[21] giving a final blow to the particle nature of light.

1.1 From Continuity to Discretisation

Hertz also performed the photoelectric emission experiment, which involved the scattering of electrons by shining radiation on metals, in 1887[22]. It was here that the wave theory of light started to receive its doubts since this phenomenon could not be explained[22]. This is because it was believed that the ejection of an electron must be dependent on the intensity of the incoming radiation. In 1905, Albert Einstein proposed a solution to the photoelectric emission, stating that light was composed of particles[23], for the experiment could not be explained otherwise. In his own revolutionary words:

> "According to the assumption to be contemplated here, when a light ray is spreading from a point, the energy is not distributed continuously over ever-increasing spaces, but consists of a finite number of energy quanta that are localised points in space, move without dividing and can be absorbed or generated only as a whole."[23]

This revived the disagreement state of the early days concerning the nature of light. Was light composed of particles or not? The question of the nature of light seemed greatly unsettled with equally valid and opposing views.

Let alone the nature of light, the interaction of light with matter was also with its difficulties. This interaction could only be explained if the nature of light was precisely known. Thus the confusion only multiplied. What is the *real* nature of light and how will this contribute to an understanding of the interaction of matter with light? This was a question that was on every physicists' mind. Such interaction was not limited to diffraction and refraction only but to many unidentified phenomena as well. It then happened that early in the twentieth century, experiments to observe the behavior of energy, matter and light involving the microscopic world provided a maze of data coming from various phenomena which was incompatible with physics. There were ideas that were very counter-intuitive and posed as profound dilemmas. One such experiment involved the radiation emitted from a black body. At the turn of the twentieth century, physicists were interested in the energy carried by light. For this, a black body seemed perfect because of its absorbing properties. The experiment determined an empirical formula for energy from high-frequency radiation given off from a black body and it seemed incompatible with an equally good representation of the low-frequency results![24] Furthermore, according to the then theory and the experimental data, there was a continuous discharge of energy as the wavelength decreased[25]. This predicted an infinite energy at very low wavelengths. The term "ultraviolet catastrophe" was introduced by Paul Ehrenfest for this singularity[26].

Should matter be then understood thoroughly? Robert Andrews Millikan, an American experimental physicist and Nobel laureate, performed his famous oil drop experiment in 1909[27] to determine the magnitude of the elementary charge. He published the results four years later in 1913[27]. In 1911, Ernest Rutherford, 1st Baron, Lord Rutherford of Nelson, of Cambridge, OM, FRS, a New Zealand-born chemist and physicist, proposed the planetary model for the

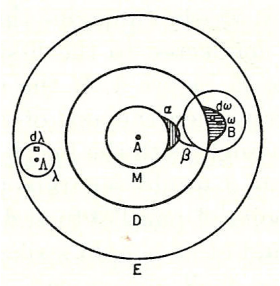


Figure 1: Ludwig Boltzmann's diagram of the I_2 molecule proposed in 1898 from Vorlesungen über Gastheorie showing the atomic "sensitive region" (α, β) of overlap

atom[28]. The famous experiment, which involved bombarding atoms to a gold foil, could point to only such a possibility. However, this breakthrough came to a halt only a little later when Rutherford himself realised that this model, according to Newton's well-grounded physics, was not stable - each orbiting electron would instantly fall into the nucleus, radiating energy as it did because of centripetal force within a fraction of a second. In other words, all of matter should have collapsed. What was to be done? Physics could not account for an understanding of light and matter. It seemed as though physics was wreaked with errors.

Ludwig Edward Boltzmann is a name that strikes as a key but hardly appreciated figure for solving the troubles[29]. He suggested that the atom world was not continuous as intuition purported it. The Austrian physicist is known to have been sent to despair for the opposition he faced against his revolutionary ideas in the fields of statistical mechanics and statistical thermodynamics. Facing severe opposition for his ideas, it was very likely that he suffered from undiagnosed bipolar disorder[30]. On a vacation in Duino, Trieste, Boltzmann hanged himself on September 5, 1906[30]. The death is well-documented[31].

In 1900, Max Karl Ernst Ludwig Planck, FRS, with a stroke of genius, perceived that introducing a quantised oscillation of the atom would correct the prediction of the experiment for a black-body[25] with his famous equation $E = h\upsilon$. Ludwig Boltzmann suggested Max Planck to use statistical methods for the blackbody radiation problem three years before he published his work[29]. Plank was, however, not content with the idea of quantising energy himself[26].

The famous Balmer series of Johann Jakob Balmer, which was introduced in 1885[32], and the generalisation of it by Johannes Rydberg, a Swedish physicist, a few years later[33] both assumed a discrete account for atomic spectra. These considerations and Max Planck's $E = h\nu$ proposal in 1900, the plum pudding model of Sir Joseph John Thomson, OM, FRS, proposed in 1904[34], and Albert Einstein's 1905 light quanta postulate prompted the Danish physicist Niels Henrik David Bohr to correct Rutherford's renowned model by making

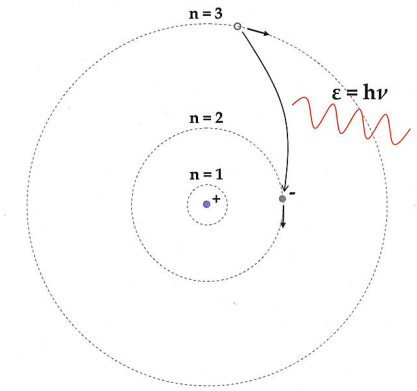


Figure 2: Niels Bohr's model of the atom proposed in 1913

the bold assumption of a discretely-levelled atom[35], saying, "we must assume that the ordinary rules of electrodynamics can not be applied ... without radical alterations"[36]. In a 1918 memoir, he wrote:

> I. That an atomic system can, and can only, exist permanently in a certain series of states corresponding to a discontinuous series of values for its energy, and that consequently any change of the energy of the system, including emission and absorption of electromagnetic radiation, must take place by a complete transition between two such states. These states will be denoted as the "stationary states" of the system.
>
> II. That the radiation absorbed or emitted during a transition between two stationary states is "unifrequentic" and possesses a frequency n given by the relation $E'-E''= hv$ where h is Planck's constant and where E' and E'' are the values of the energy in the two states under consideration

As for the nature of light, Albert Einstein's theory was confirmed by many experiments over the decade since his 1905 paper[24]. Coined by the American physical chemist Gilbert Newton Lewis in 1926[37], the word "photon" then emerged. The question of the nature of light (and matter) was finally settled by Louis-Victor-Pierre-Raymond, 7th duc de Broglie, FRS:

> "The fundamental idea of [my 1924 thesis] was the following: the fact that, following Einstein's introduction of photons in light waves, one knew that light contains particles which are concentrations of energy incorporated into the wave, suggests that all particles, like the electron, must be transported by a wave into which it is incorporated... My essential idea was to extend to all particles the

coexistence of waves and particles discovered by Einstein in 1905 in the case of light and photons."[38]

The idea had taken a new shape that shattered the philosophy of atomism. The de Broglie hypothesis had blurred the distinction between a localised particle and a continuous wave, and, in a sense, unified the ideas for radiation of Maxwell and Einstein, stating mathematically that any particle with a momentum p had a wavelength λ from $p = h/\lambda$. Max Planck's equation equally meant that a particle with a kinetic energy E had a frequency υ to which it was directly proportional, with h being the constant of proportionality. The hypothesis had profound implications - an electron in motion now was a wave smeared out in space. This wave is determined by a wave function which obey's Schrodinger's equation. The magnitude of such a wave at any instant t_0 is the probability of finding the electron in that position[1]. It is worth pointing out that the de Broglie hypothesis has been experimentally verified numerous times[39]-[46].

1.2 Philosophical Implications

These ideas seemed like simple mathematical tricks. As stated earlier, even Max Planck himself was reluctant with his new concept. It was evident, however, that physics started to fail at numerous points in the microscopic world - the so-called quantum realm, where physical properties change only in discrete amounts, or quanta, instead of a continuously varying spectra of values. A new theory was clearly needed to describe the state function of particles that was beyond the scope of the then physics. The foundations of Quantum Mechanics were thus laid that differentiated old physics, now called classical physics, from physics that was quantised. The term was coined by Max Planck in 1924[47]. Added to the difficulty of increasingly counter-intuitive ideas was the scope of approximating quantum mechanics as a limiting case to classical physics. This change may have radically effected our views of nature but that does not necessarily imply that classical physics is now superfluous - rather, the limitation and the range of validity for classical mechanics has been established.

One may observe that all of the problems discussed above were furnished with a solution, albeit ad hoc, by quantising the subject of investigation. It was only later that Quantum Mechanics emerged as a successful and rigorous theory for the behaviour of matter and energy on the scale of atoms and subatomic particles. However, the theory is not restricted to the microscopic world. Quantum mechanics has since delved into almost every aspect of the "real" world, providing a rich body of scientific principles based on a quantised view of the world. The theory hailed as a paradigm shift with its tantalising ideas in science[48]. Such disciplines include quantum chemistry, quantum biology, quantum game theory, quantum electronics, quantum computing, quantum electrodynamics, quantum cryptography, quantum optics, quantum information science, quantum field theory and quantum gravity to name a few. However, being pragmatic is not enough; Quantum Mechanics is shocking to say the least:

[1] This will be seen mathematically in the next chapter

> "For those who are not shocked when they first come across quantum theory cannot possibly have understood it" - Niels Bohr

Notwithstanding the intense knot, the departure from what was purported to be reality was not without its support and, also, antagonism. This is greatly justified by only one experiment - "the most beautiful experiment"[49] - involving the determination of the bipolar nature of matter (and hence light). When the experiment was performed, it was viewed with high suspicion. The experiment was a modified version of Dr. Thomas Young's original experiment, which involved rays of light passing through two slits and producing an interference pattern on a screen. Years later, when wave properties were understood, after de Broglie's hypothesis, electron waves were studied. A modified version of Young's double slit experiment, never carried out by Dr. Thomas Young himself[50], indicated that particles of matter (electrons) interfered with themselves and that the particles passed through both slits at the same time, even if they were fired to the slits one at a time[51]! How could a particle interfere with itself? Is matter wavy in nature or is a wave made up of matter? Where does the cause-effect relationship go? De Broglie had opened up a whole new domain for philosophers. This was not all to the experiment - the wave nature of matter was destroyed by the very act of measuring it and particles then behaved as particles and nothing more[52]. Matter (and light) were determined to be both particles and waves; they only chose to be either in different situations. This seemed like a cruel joke on the part of nature and was not very comforting. The nature of reality was on stake - everything one sees and is comforted with was on shaky foundations with embarrassing holes that, even to child, were seriously objectionable. It seemed as though the wave function reduction was extremely subjective, by being at mercy of an observer - reality is created if it is observed!

The wave function had widely differing interpretations. One of the founders of Quantum Mechanics, Erwin Rudolf Josef Alexander Schrödinger, Austrian physicist and Nobel prize winner, originally viewed the wavefunction of an electron corresponding to charge density of an object smeared out over a possibly infinite region[53]. Max Born interpreted it as simply corresponding to a probability distribution with coordinates of space and time. Simply put, the wave function is a quantum state which fully describes a quantum particle or a system of quantum particles. The development of such a quantum state showed signs of challenging credulity in an extremely troubling way. John Stewart Bell is known to have mocked this situation: "I am a quantum engineer but on Sundays I have principles."[54]. To see what it is about quantum mechanics that has troubled physicists and philosophers alike, one must be equipped with a few technical words. Embarking on such an overwhelming expedition will then provide great insight. *Realism* states that reality is independent of a conscious mind[6]. For our purpose, *objectivity* may be considered synonymous with realism. This is contrasted to *subjectivity* and *idealism*. The word "conscious" has been taken in its broadest, undefined sense. *Positivism* asserts that all meaningful statements are either analytic or conclusively verifiable by direct observation[55]. *Determinism* is defined as a completely ordered idea, in which an event can be

determined as an effect of its past causes[6].The *Principle of Locality*, in Einstein own words, exerts the relative independence of two objects in space, where it is proposed that an influence on one object will leave the other unperturbed[56]. An *instrumentalist* is a person who believes that a certain idea is true if it has practical applications[6].

In quantum mechanics, an object exists in a superposition of states. The probability amplitudes of each state interfere with themselves and each other. For example, an atom with two levels represented by $|excited\rangle$ and $|ground\rangle$ will exist in the superposition $\frac{1}{\sqrt{2}}|excited\rangle+\frac{1}{\sqrt{2}}|ground\rangle$ i.e. simultaneously. To mark the idea, Erwin Schrodinger proposed a *gedanken* in 1935[53] in which he proposed a cat in a box trapped with an atom that decays after an hour with 50% chance. The decay of the atom releases a poisonous gas, thus killing the cat. An hour later, the cat exists in a superposition state

$$\frac{1}{\sqrt{2}}|alive\rangle_{cat} \otimes |stable\rangle_{atom} + \frac{1}{\sqrt{2}}|dead\rangle_{cat} \otimes |decayed\rangle_{atom}$$

The cat is both dead and alive!

A measurement of a quantum state in a linear superposition will yield in only one state. What state will be yielded? This one fundamental question is of primary importance in quantum mechanics. Quantum Mechanically, each state is a possible result. The results may not even be the same for identical systems. This is where quantum mechanics has taken its flight from determinism. The outcome is dependent on a probability distribution (the wave function) of such a system. Even the probability may not be continuous. This irreversible change caused by measurement is called a wave function collapse. Consider a ball on the left side of a toy-hill, trying to cross it over. If the initial kinetic energy provided to the ball is greater than the potential energy needed to climb the toy-hill, the ball will be on the right side of the hill, no matter how many times the experiment is repeated. On the other hand, if a quantum particle tries to pass through a door, the wave function will assign probabilities to the position of the particle. This means that the particle may be present behind the door, in front of the door and even in the door. No matter what the initial velocity of the particle is, it will stay in a superposition of these three states. If an objective reality of quantum mechanics is considered, the particle present in all three places and places between them simultaneously.

One simple example of wave function collapse is the measurement performed in the modified version of Young's double slit experiment. It can easily be seen that this idea has opened a large world of theories and possibilities. A plausible thing to do would be to turn to the definition of measurement. To date, there is no universal consensus on what constitutes a Quantum Measurement[57]. This is because the measuring device must also obey the rules of quantum mechanics and must, therefore, be subject to limitations. It is said that the state decoheres with the environment. This is true in the Copahengen Interpretation of Quantum Mechanics. This interpretation was formulated by Bohr and Werner Karl Heisenberg in 1927, while they were collaborating in Copahengen[58]. According to them, to ask the state of a system before measurement is meaningless.

The state only chooses one value after it is measured. Reality *is* created when it is measured! This prompted Einstein to remark, "Is the moon there when nobody looks?"[56] This revived a state of confusion in the days of Bishop George Berkeley whose *esse est percipi* ("to be is to be perceived") proposed that if a tree falls in a forest and no one is around to hear it, it does not exist.[59]

This interpretation is very close to the objective collapse theory of quantum mechanics, except that the latter believes in the objective reality of the wave function. In this theory, a collapse will occur in a indeterministic way i.e. random way or when a physical threshold is reached. The role of measurement and a conscious observer are discarded. This last remark opens room for an important discussion: the role of consciousness. In fact, there is one complete interpretation based on a wave function collapse caused by a conscious observer. This was purported by John von Neumann in 1932[60]. He reasoned as follows: the complete world may be regarded as one wave function. Since something "outside" the wave function was needed to measure it and cause a collapse, he concluded that the collapse was due to a conscious observer. This raises many critical questions. Can a cockroach cause a wave function to collapse? An earthworm can split into two earthworms; can they have an entangled conscious or do they independently cause a wave function to collapse? If, on the other hand, all our conscious was grouped together, were we then the cause of our own wave function's collapse? Where does lucid mysticism, to use the words[61] of the Austrian physicist and one of the founding father of quantum mechanics Wolfgang Ernst Pauli, differ from physics?

In another interpretation, the wave function simply takes one possible value, asserting that the wave function has an objective reality. This interpretation, however, denies that a collapse ever happens. Instead, it asserts that the wave function splits into many possibilities in different worlds, not observable to each other. Thus, Schrödinger's cat branches off in history upon a measurement. The objective reality of the wave function exerts that the cat is both alive and dead in the box, existing in a superposition of states. The "alive" and "dead" cats are in different branches of the universe, both of which are equally real, but which cannot interact with each other. Are there infinite worlds? The answer, the author believes, lies in the tastes of the reader. Regardless of how unconvincing the argument may sound, this approach does win determinism back.

Another interpretation of quantum mechanics utilises the concept of the mathematical probability distribution. It states that the wave function, without an objective reality, can only be applied to an ensemble of particles and not one individual object. This is called the ensemble interpretation[62]. The laws of classical probability then come into play. The wave function can not exist in more than one states for one object and thus the wave function can never really be reduced. The very foundation of this interpretation has been the cause of criticism[63].

Perhaps the closest any interpretation got to determinism was the de Broglie–Bohm theory, called the pilot wave theory, in 1952[64]. This is a theory because it involves the radical idea that the wave function actually guides the trajectory of the particles. The wave function is given a specific force, called the

quantum force in the form of hidden variables, which interacts with all other classical forces. It may seem paradoxical to work with hidden variables. Bohm-de Broglie worked on a theory that accounted for trajectories in the form of hidden variables. The wave function then pointed to trajectories a particle could take. This theory is completely deterministic. It was not well received because it was seen as a step backward from the, perhaps, liberating ideas of quantum mechanics[64].

There is another problem associated with Schrodinger's cat state. If the atom has decayed, we know that the cat is dead. If the cat is alive, then we know that the atom has not decayed. Any state that exists in a superposition, no matter how large their spatial separation is, the measurement of one state will reveal the state of the other. This is true if the states are entangled, like Schrodinger's cat state. In Schrodinger's own words, entanglement happens "when two systems, of which we know the states by their respective representation, enter into a temporary interaction due to known forces between them and when after a time of mutual influence the systems separate again. They no longer can be described as before but by endowing each of them with a representative of its own. I would not call [entanglement] one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought"[65]. In other words, even if two objects are spatially separated by a distance of light years, information about such an object can travel faster than light. It suffices to speak of the importance of entanglement through variations like "spooky action at a distance"[56] and "contagious magic"[66]. Has physics gone this bizarre? It may be that such imperfections in a theory may be the result of a flaw in our experiments (indirectly, our measurements) or of our limited knowledge.

The latter idea was fully developed as a theory of hidden variables in the famous Einstein-Podolsky-Rosen Paradox in 1935[67], which even went further to state that Quantum Mechanics was incomplete. It seems as though the man behind the idea of a photon was now bent on bringing determinism back. The EPR argument, carrying the belief that "the Old One does not play dice"[68], imagined a particle decaying into two particles speeding off in different directions with exactly the same spin magnitude but different directions. If the position of one particle is measured without any way disturbing the other particle, the position of the other may be determined. Using a similar reasoning, the momentum of both particles may be determined. In order words, if the principle of locality were to hold, then there is no way that one could have information travelling faster than the speed of light. Thus, the idea essentially states that if two measurements are made of a decaying particle, one measurement for each by-product, both the position and momentum of the quantum particles may be determined by using the laws of the conservation of energy. This is in direct contradiction to the Hiesenberg Uncertainty Principle which says that both the position and momentum (non-commuting operators) can not be both simultaneously determined. Although the assumptions in the EPR Paradox did not satisfy all of the postulates of quantum mechanics, yet the idea of an incomplete theory was still frustrating for proponents of Quantum Mechanics because

the EPR Paradox did not receive a satisfactory reply. The idea had gained wide attention; intuition could now rest unchallenged. If the 1935 paper was wrong, then so was the law of conservation! However, the merry days for realism, determinism and locality were over when John Stewart Bell published his famous no-go theorem in 1964, called Bell's inequality, that resolved the EPR paradox[69].

Theorem 1 *No physical realistic and local theory can agree with all of the statistical implications of Quantum Mechanics[70]*

Proof. Let's say Alice and Bob each posses two particles, whose state they can measure with two detectors each. The detectors are arranged at random. If these particles are photons, we could have detectors arranged to detect vertical and horizontal polarizations. If we have electrons, we could arrange to have a $\pm$ spin 1/2 state with a corresponding detector. Since a general treatment is being considered, this theorem is within the reach of an experimental realisation. For now, the exact configuration is not important. What is important is the assumption that the particles possess two arrangements each which can be replenished for as long as necessary, that these arrangements have an objective reality before they are measured and that a measurement will only reveal to the observer a property of that particle. The moon must exist even if we are not looking, after all! Each measuring device has a randomly chosen arrangement, such that the one arrangement does not alter or influence the other Thus, if there are two possible measurements, then one result would occur with a probability of 1/2. Let's call the arrangements of the particles Alice has a and c and those of the particles Bob has as b and d. When a randomly chosen measurement is made, a value of, say, 1 is assigned to it whereas a corresponding score of -1 is applied if the measurement reveals another arrangement of the particle. The measurements performed by Alice and Bob must be made at exactly the same time so that in now way do they influence each other. It is further assumed that in no way does one particle change the other when a measurement is made. This last sentence combines to give us, what is called, local realism. Then, the following table lists the possible outcomes:

Particle a	Particle c	Particle b	Particle d
1	1	-1	-1
1	-1	-1	1
-1	1	1	-1
-1	-1	1	1

This table obeys the equality

$$ab + cb + cd - ad = \pm 2$$

If the number of measurements is gradually increased, we can take the average[2] of the measurements. This gives us the CHSH inequality

$$\langle ab \rangle + \langle cb \rangle + \langle cd \rangle - \langle ad \rangle \leq 2$$

[2] See appendix

If classical intuition of locality were true, this equality would hold. In quantum mechanics, this is not the case. Imagine an entangled Bell-state[3] $\frac{|01\rangle-|10\rangle}{\sqrt{2}}$. Now, this state is given to Alice and Bob. Alice only performs a measurement on a and Bob measures b. Suppose the measurement basis change to $|0\rangle\langle 0| - |1\rangle\langle 1|$ and $\frac{|1\rangle\langle 1|-|0\rangle\langle 0|-|0\rangle\langle 1|-|1\rangle\langle 0|}{\sqrt{2}}$ for a and c, respectively and $|0\rangle\langle 1| + |1\rangle\langle 0|$ and $\frac{|0\rangle\langle 0|-|1\rangle\langle 1|-|0\rangle\langle 1|-|1\rangle\langle 0|}{\sqrt{2}}$ for Bob's b and d, respectively. These are projection operators that can act on a certain state vector[4]. Then, $\langle ab\rangle = \langle bc\rangle = \langle bd\rangle = -\langle ad\rangle = \frac{1}{\sqrt{2}}$. These values do not satisfy the CHSH inequality. This simply means that one of our assumptions is incorrect. The alert reader may now understand why unnecessarily listing each aspect of our accepted version of reality was important because this very simple mathematics has revealed otherwise. Now, what is it that is not in the flavour of Quantum Mechanics; locality, parts of reality or reality itself? ■

Quantum Mechanics has killed locality and determinism because it slew positivism. Why does Quantum Mechanics do that? What actually is it to the meaning and mystery of quantum mechanics? Perhaps there is no need to interpret mathematics at all, for, all of the above implications stem from a realist's view of mathematical formalism. A *realist* is a mathematician who believes that mathematics has a correspondence to external reality. Albert Einstein was a realist[71].A *Platonist*, on the other hand, is a mathematician who believes that mathematics draws from a higher, unchanging and ultimate reality called a Platonic heaven - a place where abstract ideas reveal themselves so that reality may be approximated to them. The Austrian logician, mathematician and philosopher Kurt Friedrich Gödel, on the other hand, was a Platonist and also an idealist[72]. This greatly simplifies many debates since the mathematics of quantum mechanics derives its ways from a transcendental reality, which is beyond the scope of humanity to interpret. This may seem like dabbling away with interpretations. On the other hand, to struggle with these counter-intuitive notions appears to be a physicists passion - they can not do without an interpretation. The Copahengen Interpretation has been chosen as the standard interpretation (with due disagreement, of course). The division of physicists may be noted for its connection in the long "idealist vs. realist" debate. As a comment, it maybe added that the collapse of a wave function has also been argued to be nothing more than a mathematical entity[73]. This argument stems from a *formalist*'s view of mathematics. It may also be tempting to take the *instrumentalist*'s point of view with quantum mechanics which weighs the truth of knowledge according to its usefulness. Such an interpretation finds many supporters, but to them, the opposition simply states that theories may work for the wrong reasons. The point of concern of this section, however, was on the phenomenology and formalism of quantum mechanics - the mathematics

[3] a state like the Schrodinger cat state represented, for convenience, as $\frac{1}{\sqrt{2}}|1\rangle_{cat}\otimes|0\rangle_{atom} + \frac{1}{\sqrt{2}}|0\rangle_{cat}\otimes|1\rangle_{atom} = \frac{|10\rangle+|01\rangle}{\sqrt{2}}$. The cat and the atom are not necessarily sole candidates for this entangled state and may be replaced by any particle.

[4] See appendix

and observation. It is on the nature and limitations of knowledge and the relationship of reality with the being that physicists are divided and thus, a conclusion is ultimately left to correspond to the taste of the reader to choose a utopia of ideas over realism.

The length of the thesis only allows partial window to be opened. For a vista, the curious reader may refer to references [56], [65] - [73], which detail an account on the ideas and challenges posed by Quantum Mechanics. For now, the reader should excuse the author for the following quote by Nathaniel David Mermin: "Shut up and calculate!"[74]

1.3 Technological Advances

Quantum Mechanics has enjoyed extreme precision and success. Being the best candidate for the small world beyond the naked eye, Quantum Mechanics explains why, which and how waves emerge from atoms. It is also widely used by chemists to explain inter-molecular properties and structure[75]. Mathematically, as will be seen later, the theory is well-grounded. Of the many gifts classical physics has given society, there is one an important utility viz. radio communication, revolutionising many aspects of our lives. No body would doubt that during World War II, radio communication must have been very crucial. What classical physics does not fully tell us is how these waves interact with metals. Sir Rudolf Ernst Peierls, CBE, Flex Bloch and Alan Wilson filled this gap and developed a band theory of solids using the principles of Quantum Mechanics. Any engineer familiar with semi-conductor physics based on this band theory would not deny the impact of this theory in micro-electronics.[76]. Quantum Mechanics is, thus, not just a mathematical curiosity. To make the point hit home, two latest developments will be discussed viz. the Bose-Einstein Condensate and the Recurrence Tracking Microscope.

To explain the Bose-Einstein Condensate, it is important to first talk about matter. Matter exists in different states, dependent on their temperature. Imagine having a state of matter different from solid, liquid and gas at temperatures with in a fraction of absolute zero that would cause the particles of matter to fall or condense beyond anything ordinary. A Bose–Einstein condensate is such a state of matter. It consists of a dilute gas of weakly interacting force carrier particles which obey certain statistics. The particles are confined in an external potential so as to isolate them. This has the effect of revealing quantum mechanical properties. The atoms transform into waves, spreading out in space, even though they are "sitting" in rest, interfering with each other, losing their individual identity and combining into one complete wave packet being everywhere at once! This is quantum mechanics at hand with extremely blended atoms, which form a super-atom. Quantum Mechanical effects can now be studied on an easier level. This condensed state of matter was predicted theoretically by Satyendra Nath Bose an Indian mathematician and physicist, in 1924, who collaborated with Einstein for his theory. Experimental hindrences took their toll on the quest for absolute zero, until seventy years later, when the first Bose-Einstein Condensate was observed. This achievement was crowned

with the 2001 Nobel Prize in Physics.

The father and son duo William Lawrence Bragg and William Henry Bragg proposed what is now called Bragg's Law in 1913. The equation they proposed gives the angles of coherent and incoherent waves scattered off crystals. A diffraction pattern is observed and this allows for the structure of crystals to be probed. The incident rays disturb the electron cloud of the crystal. This shift of charges causes the atom to radiate the same incident radiation but in a different direction. The energy and hence wavelength of the incoming radiation is preserved; only a change in direction is recorded. For this to take place, the wavelength of the incident radiation must be comparable with interatomic distances. This diffraction can be completely explained using Maxwell's equations. However, any impurities in the atoms are also recorded. Often, the resolution is not of an acceptable extent. If the atoms are themselves dynamic, the diffraction becomes frustratingly difficult. There have been many advances in this field of microscopy. A probe proposed by Dr. Farhan Saif[77] is based on quantum recurrence phenomenon The phenomenon simply states that a closed quantum system will return to its initial state, hence the word recurrence. In the microscope proposed, a cloud of cold atoms like the Bose-Einstein Condensate bounce elastically off a special mirror. This mirror is connected to a cantilever that records the bounces. The cantilever is influenced by the dynamical nature of the atoms. The recurrence of the atoms bouncing off of the atomic mirror is easy to calculate if the height of the atoms is known. If the surface under investigation is dynamic, quantum recurrences are recorded, keeping in mind both the periodicity of the bouncing atoms and the surface under investigation. This determines the structure of the surface.

Quantum Mechanics has a rich theory developed in the areas of cryptography, superdense coding and teleportation to name a few. Another quantum theory employed in technology is that of computation. Classical computers, modelled by a Turing Machine, use a step-by-step sequence to execute algorithms. This is a hindrance in many ways. For example, in many cases, the best-known search algorithm is a brute search. Quantum Mechanics employs its power of parallelism to execute many commands in parallel. This yields an impressive exponential speed-up. One major reason to opt for a quantum computer is because of Moore's Law, which states the computation power has been increasing ever since the invention of a silicon chip because of a growing demand of computation power. This is contrasted with a decrease in the size of the chips. This can not go on forever as the size of chips are reduced to the length of atoms - quantum mechanical effects will dominate. Since quantum mechanics has an inherent uncertainty, computation will become probabilistic. This is bad news since this may mean a severe computation crisis and all that is linked to computers, which is not limited to banks, mobile phones and TVs. To avoid losing information and computation, quantum computation developed as an alternative. This computer model based on the principles of Quantum Mechanics has been well-defined with a rich theory. All that remains is for engineers to build a physical quantum computer that will perform all that is limited on a classical computer.

The harmonic oscillator quantum computer is the simplest known example of an implementation of a quantum computer. This technique uses the different discrete energy levels of an atom, say. Nuclear Magnetic Resonance, Ion traps and photon quantum computers are also other well-known techniques. It is worth mentioning that the Bose-Einstein Condensate has relevance to quantum computation - instead of using lasers for computation, one could employ the Bose-Einstein Condensate, which produces equally coherent waves or atomic lasers. These coherent atoms in flight are used for computing. The state of an electron are also employed in quantum computation. Consider an electron's transition from one state to another as it exchanges an energy packet in the form of a photon, if the frequencies match with the incoming radiation. To make this experimentally feasible, atoms are shot in a cavity, with the detuning between the cavity and the incoming atoms minimised. This principle is exploited in engineering quantum algorithms using cavity quantum electrodynamics. This method thus employs the internal states of an atom for computation, which is what has been expounded in this thesis.

1.4 Overview of work in thesis

A resonant atom-field interaction is not all to the technique employed in engineering a quantum algorithm in this thesis; a modified approach, based on the theory developed in [78] has also been used. Here, the detuning is made large enough so that the interaction of the cavity with the atoms is off-resonant. Instead, the atoms receive a kick that is not sufficient for a transition but causes a change in its momentum states, also called external states, of the incoming atoms. Energy levels represent different quantum bits, called the internal states. If the energy levels are not altered during an interaction with any cavity, i.e. there is no resonance of the atomic energy differences and cavity's field, then external states of the atom are taken for quantum bits. An algorithm called the Deutsch Algorithm has been engineered using such techniques. This algorithm completely resolves, in a deterministic way, an unknown Boolean function, provided that it has a certain properties. In engineering the algorithm using quantum optics, a series of cavities are positioned to interact with three atoms, one at a time. Atomic waves are passed through each cavity. The arrangement is such that the sequence implements the algorithm. It will be seen later in this thesis that the choice of representation using internal and external states is not arbitrary. Different interaction times are selected in each cavity to perform unitary transformations of the external states. The preparation and measurement of such a state, however, has not been discussed. Previously, the algorithm has been engineered by Shi-Biao Zheng[79] using the internal states of atoms. This thesis modifies this approach to use the external states of atoms by using techniques of Bragg diffraction. This technique is in the scope of current technological standards.

We now turn to cover the mathematical formulations necessary for quantum mechanics.

Hilbert space is a big place - Calrton Caves

2 Mathematical Preliminaries

The aim of this section is to make the thesis as self-contained and elaborate as possible. There are several items of nomenclature and notation which may be a cause of confusion amongst themselves because of their similarity. In order to avoid that, details of each definition have been explicitly detailed in the appendix. It may be noted that the something as complex of as a Hilbert space for doing Quantum Mechanics starts with the simple definition of a set and function. As beautiful as this idea may seem, it may also be added that this mathematics is not without its difficulty. Maxwell's far-fetched equations are compatible with only the real number system and a differential operator. On the other hand, a complex Hilbert space is needed for Quantum Mechanics. Instead of differential and integral operators, there is a rich array of operators to choose from. Linear Algebra is fully expounded. There are many different fields that are in play together to make a working system for quantum mechanics.

2.1 Momentum, position and energy operators

Such lengthy and fancy mathematics has been introduced in order to rigorously state the momentum, energy and position of a quantum state. Classically, momentum and position are dynamical variables. In Quantum Mechanics, dynamical variables are replaced by operators of such observables. This complete operator theory applies naturally to these observables. The proper methodology of their employment has been discussed in the postulates of quantum mechanics. The position operator $\hat{\mathbf{r}}$ is simply defined as $\hat{\mathbf{r}} := \mathbf{r}$. The motivation for this definition becomes clear when one considers $\mathbf{r}$ as an averaged position of a system. The momentum operator is defined as, in spatial coordinates, $\hat{\mathbf{p}} = -i\hbar\nabla$ where $\hbar$ is Planck's constant and ∇ is the nabla operator. Under a suitable transformation, the momentum operator is equivalent to the position operator. This implies that the both operators are equivalent in different spaces. In one space, however, the operators still do not commute:

Proof. Let ψ be any state. Then,

$$\begin{aligned}
[\hat{\mathbf{r}}, \hat{\mathbf{p}}]\psi &= \hat{\mathbf{r}}\hat{\mathbf{p}}\psi - \hat{\mathbf{p}}\hat{\mathbf{r}}\psi \\
&= -\mathbf{r}\left(i\hbar\nabla\left(\psi\right)\right) + i\hbar\nabla\left(\mathbf{r}\left(\psi\right)\right) \\
&= i\hbar\psi\nabla\mathbf{r} \\
&= i\hbar\psi
\end{aligned}$$

Since this holds for any ψ, it can safely be said that $[\hat{\mathbf{r}}, \hat{\mathbf{p}}] = i\hbar$ ■

The energy operator is defined as $\hat{\mathbf{E}} := i\hbar\frac{\partial}{\partial t}$. Since energy and time have the same relation in special relativity as momentum and veloctity, therefore .$[\hat{\mathbf{E}}, \hat{\mathbf{t}}] = i\hbar$. The problem with $\hat{\mathbf{t}}$ is in its definition; $\hat{\mathbf{t}}$ can not be defined as $\hat{\mathbf{t}} = t$ since its Fourier transform is not possible because there is no "negative time" and,

consequently, can not correspond to energy. However, Quantum Mechanics does not treat time and position on equal footing as special relativity. In Quantum Mechanics, time is just a parameter, useful for labelling the wavefunction and there is no such operator t that would give us a commutativity with $\mathcal{H}$ of $i\hbar$.

Proof. Suppose the contrary and say $[\mathcal{H}, \hat{\mathbf{t}}] = i\hbar$

where ψ is any eigenstate with eigenenergy E.

$\Rightarrow e^{-ic\hat{\mathbf{t}}}\mathcal{H}e^{ic\hat{\mathbf{t}}} = \mathcal{H} - c$ for any $c \in \mathbb{R}$

Now, if $\mathcal{H}\psi = E\psi$, then

$e^{-ic\hat{\mathbf{t}}}\mathcal{H}e^{ic\hat{\mathbf{t}}}\psi = (E-c)\,\psi$

$\Rightarrow E$ has a continous energy spectrum in $\mathbb{R}$

$\Rightarrow E$ is unbounded ■

In analogy to $\Delta p \Delta x \geq h$, we must have $\Delta E \Delta t \geq h$ but then there is an inherent difficulty in addressing Δt. If it corresponds to the interval a state can exist, then, heuristically, a state that cannot exist in a short span must have an indefinite amount of energy, and contrariwise. When events transpire at a shorter interval, there is a large uncertainty associated with the energy of the events. Is the entropy principle voilated? Indefinite energy has to be borrowed and returned. This is consistent, as long as the total energy of the universe is not an exactly known parameter at any time. In conclusion, one might suggest that the energy of a quantum system cannot be determined with one history; this is in agreement with the spirit of Quantum Mechanics.

2.2 Hamiltonian

Of particular importance are the position and momentum operators because of their clear fundamental importance even in classical mechanics to determine the total energy (kinetic and potential) of a system. Classically, the Hamiltonian of a system is described as the total energy of a system. The determination of a Hamiltonian following from the principle of least action can be put aside for now. What matters is that the Hamiltonian carries its classical definitions over as the sum of the kinetic energy T and potential energy V of the system. In Quantum Mechanics, however, the Hamiltonian is an operator that acts on a state. By definition, the Hamiltonian is assumed to be Hermitian. It differs from its classical counterpart by being an operator instead of an element of a field. This is because kinetic and potential energy are formed so by using operators of momentum and energy defined on the Hilbert space. $L^2(-\infty, \infty)$[5]. This is the state space for Quantum Mechanics.

2.3 Uncertainty Principle

In 1927, Werner Heisenberg stated that for any two non-commuting observables, $\hat{A}, \hat{B}$, $\triangle A \triangle B \geq \frac{1}{2}\left|\left\langle [\hat{A}, \hat{B}] \right\rangle\right|$[6]. This mathematical assortment is of great

[5] See appendix

[6] See appendix

value and is the core reason why quantum mechanics exists. Since the observables do not commute, the eigenvalues of both the variables can not be determined simultaneously. In physical language, this means that two non-commuting observables can not be measured simultaneously. From the above theorems, $\triangle \mathbf{r} \triangle p \geq \frac{\hbar}{2}$ and $\triangle E \triangle t \geq \frac{\hbar}{2}$, Both the variables can not both be determined with a limit less than $\frac{\hbar}{2}$. The more accurately one tries to measure one observable, the other expands its uncertainty. A simple algebraic manipulation reveals that if the energy of a system is determined with complete accuracy, then $\triangle t$ becomes very large Similarly $\triangle \mathbf{r} \rightarrow \infty$ as $\triangle p \rightarrow 0$. This is where quantum mechanics departs with classical mechanics: the momentum and velocity of a body can not both be determined with complete accuracy. This departure marks the core of the theory of quantum mechanics[80].

The notion of uncertainty deserves some elaboration as this entails a lengthy discussion, as was hinted above. Uncertainty is normally referred to as a lack of knowledge. To an experimentalist, it has to do with inaccuracy that is garnered through the execution of the experiment or even in the definition of a quantity. In quantum mechanics, it is normally understood as a statistical spread in an collection of similarly prepared systems. Being an intrinsic part of quantum mechanics, uncertainty leads us to question the following: are these restrictions on the experiments that can be performed? If that is the case, there are inherent limitations on the information that can be gathered, which is not very comfortable for many. Is the relation an inherent limitation of ideas that can be borrowed from classical mechanics? That would mean a whole new theory for quantum mechanics is needed. Finally, is this the way quantum mechanics simply is - indeterministic as the Copahengen interpretation puts it? In fact, many choose to call the Hiesenberg's inequality as simply a relation instead of a principle. This stems from one question: is the aim of physics limited to describe observable phenomena? Certainly, many physicists themselves even dislike the idea of multiple universes and parallel dimensions. This leads into a debate of ontology which is beyond the scope of the thesis. All such questions are left to the reader to explore. Interested readers are referred to [6] for fundamentals of ontology and to [81] for a rich discussion on the philosophy of quantum mechanics.

2.4 Postulates of Quantum Mechanics

There are a number of postulates of quantum mechanics that, if found not contradicting experimental results, may be assumed as axioms. They are, thus, the starting point for all of the mechanism for quantum mechanics. Essentially, they list the rules for determining a quantum state which normally depend on some parameter(s). Below are listed four axioms usually taught at undergraduate level[82]:-

1. The state of a system is described by a complex-valued wave function in $C^1(\text{-}\infty, \infty)$, normally denoted by ψ. The fact that it is a function implies that only one value must correspond to each $(\mathbf{r}, t)$. $\psi(\mathbf{r}, t)$ describes the

state in terms of a probability amplitude i.e. $|\psi(\mathbf{r},t)|^2 = \overline{\psi(\mathbf{r},t)}\psi(\mathbf{r},t)$ is the probability density function. Thus,

$$\int_{-\infty}^{\infty} |\psi(\mathbf{r},t)|^2 \, d\mathbf{r} = \int_{-\infty}^{\infty} \overline{\psi(\mathbf{r},t)}\psi(\mathbf{r},t) d\mathbf{r} = \mathbf{1}$$

This normalisation condition simply states that the particle has to be somewhere! It is important to note that the integral remains the same even if it is multiplied by a complex number of absolute value 1.

2. Every physically measurable, dynamical variable (observable) has a corresponding Hermition operator. Such an operator will be of the form $T : H_2 \longrightarrow H_1$, where H_1, H_2 are Hilbert spaces. An observable is a quantity which can be measured. Examples include position, momentum and energy. The eigenvectors of such a Hermitian operator form a complete basis of the state vector.

3. The possible results of each observable are the eigenvalues of the corresponding operator. Normally, the eigenvectors are called eigenstates. Since observables are Hermitian, the eigenvalues of this observable are real. It may seem strange to state a measurement postulate in order to rigorously define how information can be retrieved from a system but in quantum mechanics, the problem of measurement has far-reaching implications. A window to such possible implications has already been seen in the introductory chapter. This postulate is of special interest because a measurement on a state in a superposition of states will yield different results each time it is evaluated! Of interest is also the state of the system after a measurement is made. In classical mechanics, a measurement generally has no effect on a system. In quantum mechanics, measurement is an interesting phenomenon because the very act of measuring causes an irreversible change in the system. This is fancifully stated as "destroying the wave function".

4. A projection operator $\hat{P}_i$ that acts on a state in order to measure an observable reduces the state to its corresponding eigenvector. A rich theory of measurement called the projective measurements or sometimes called the von Neumann measurements, has been developed. The projective measurement approach uses simple projection operators, described in Chapter 2. It may perhaps be no surprise that this postulate accounts for the wave function reduction. The vector that remains after the action of a projection operator is then normalised to meet postulate 1. In functional language notation, the relation

$$\frac{dP(\mathbf{r})}{d\mathbf{r}} = \frac{\overline{\psi(\mathbf{r},t)}\psi(\mathbf{r},t)}{\int_{-\infty}^{\infty} \overline{\psi(\mathbf{r}_0,t)}\psi(\mathbf{r}_0,t) d\mathbf{r}_0}$$

calculates the probability of finding a particle within a small region $d\mathbf{r}$ after the action of an observable[7]. In simpler terms, this postulate simply asks us to consider the following system:-

$$\alpha|\psi\rangle + \beta|\phi\rangle$$

where $|\alpha|^2 + |\beta|^2 = 1$. Upon measuring, the system is forced to change to either $\alpha|\psi\rangle$ or $\beta|\phi\rangle$ with probabilities $|\alpha|^2$ and $|\beta|^2$, respectively. This is irreversible. Thus, in order to be able to determine the values of α and β, one measurement does not suffice since one measurement will give way to a collapsed wave function. The expected result of a measurement is described by a continuous or discrete probability distribution, depending on the state that is being measured. A measurement is made with a device called the ancilla, which is a part of the environment. A quantum system thus loses its information to its environment. Since probabilities must have a sum equal to 1, correspondingly, a complete set of orthogonal projection operators are those for which $\sum_i \hat{P}_i = I$ which simply means that at least one $\hat{P}_i$ must come into play during a measurement. For the above system, we have $\hat{P}_1 = |\psi\rangle\langle\psi|$ and $\hat{P}_2 = |\phi\rangle\langle\phi|$ for $\hat{P}_1 + P_2 = I$ if $\langle\psi|\phi\rangle = 0$

5. The time evolution of a state is determined by the Schrodinger equation.

$$\hat{H}\left(\psi\left(\mathbf{r},t\right)\right) = \hat{E}\left(\psi\left(\mathbf{r},t\right)\right)$$

Here, $\hat{H}$ is the Hamiltonian of the system in consideration whereas $\hat{E}$ is the energy of the system. This may or may not be an operator. This crucial postulate is used to determine how the state $\psi(\mathbf{r},t)$ evolves over time and space. There are different methods to determine the state function if the total energy of the system is known. One such method called the probability amplitude method has been utilised.

2.5 Pauli Operators

For an atom's ith energy level, represented by $|i\rangle$[8], a transition from i to the jth energy level $|j\rangle$ is represented by $\sigma_{ij} := |j\rangle\langle i|$[83].

1. $[\sigma_{ji}, \sigma_{kl}] = \delta_{ik}\sigma_{jl} - \delta_{lj}\sigma_{ki}$
2. $[\sigma_-, \sigma_+] = -\sigma_z$
3. $[\sigma_-, \sigma_z] = 2\sigma_-$

[7] A few text books replace this postulate by accounting for the expectation value of a variable. In this thesis, this is accounted for in the definition of the probability density function in the appendix.

[8] From here on, the Dirac notation will be used because of its flexibility. See the appendix for a review of the notation

Proof. $[\sigma_{ji}, \sigma_{kl}]$
$= |j\rangle\langle i|k\rangle\langle l| - |k\rangle\langle l|j\rangle\langle i|$
$= |j\rangle\delta_{ik}\langle l| - |k\rangle\delta_{lj}\langle i|$
$= \delta_{ik}|j\rangle\langle l| - \delta_{lj}|k\rangle\langle i|$
$= \delta_{ik}\sigma_{jl} - \delta_{lj}\sigma_{ki}$

If the atom has only one excited level $|a\rangle$ and one ground level $|b\rangle$, then, the raising and lowering operators are defined, respectively, $\sigma_- := |b\rangle\langle a|$ and $\sigma_+ := |a\rangle\langle b|$. The Pauli inversion operator is defined as $\sigma_z := \sigma_{aa} - \sigma_{bb}$. Thus, 2. can be verified as follows:
$[\sigma_-, \sigma_+]$
$= \sigma_-\sigma_+ - \sigma_+\sigma_- = \sigma_{ba}\sigma_{ab} - \sigma_{ab}\sigma_{ba}$
$= |b\rangle\langle a|a\rangle\langle b| - |a\rangle\langle b|b\rangle\langle a|$
$= |b\rangle(1)\langle b| - |a\rangle(1)\langle a|$
$= |b\rangle\langle b| - |a\rangle\langle a|$
$= -\sigma_z$
For 3., $[\sigma_-, \sigma_z]$
$= |b\rangle\langle a|\left(|a\rangle\langle a| - |b\rangle\langle b|\right| - \left(|a\rangle\langle a| - |b\rangle\langle b|\right)|b\rangle\langle a|$
$= |b\rangle\langle a|a\rangle\langle a| - |b\rangle\langle a|b\rangle\langle b| - |a\rangle\langle a|b\rangle\langle a| + |b\rangle\langle b|b\rangle\langle a|$
$= |b\rangle\langle a| + |b\rangle\langle a| = 2\sigma_-$ ■

It is inherently assumed that the basis vectors $|b\rangle$ and $|a\rangle$ are orthonormal.

2.6 Entanglement

One striking feature of quantum mechanics is the concept of entanglement with far reaching consequences. One such consequence has already been observed in Bell's theorem. Entanglement simply means that a state shares certain properties with another, becoming correlated even if they are separated by a distance of light years. Changing one state will cause a change in another. One neat application of this is in the technique of superdense coding. For two qubits, entangled states, called Bell states, are[84]

$$
\begin{aligned}
|\beta_{00}\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
|\beta_{01}\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}} \\
|\beta_{10}\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
|\beta_{11}\rangle &= \frac{|01\rangle - |10\rangle}{\sqrt{2}}
\end{aligned}
$$

Mathematically, two states are entangled if they can not be factorised into individual components - notwithstanding spatial separation, such states still act as one, without an individual identity.

Entanglement is not just a mathematical curiosity. The external states of an atom can be entangled[85] Also, two different modes of a cavity may get

entangled[86]. Photon states can also be entangled using the Bragg regime in cavity QED[87]. Even Bell states may be generated via atom interferometry[88]. For more details, see a complete thesis in [89].

"What kind of computer are we going to use to simulate physics? ... Can physics be simulated by a universal computer? ... Can you do it with a new kind of computer - a quantum computer? ... Can a quantum system be probabilistically simulated by a classical (probabilistic, I'd assume) universal computer? In other words, a computer which will give the same probabilities as the quantum system does. If you take the computer to be the classical kind I've described so far ... and there're no changes in any laws and there's no hocus-pocus, the answer is certainly no!" - Richard Feynman

3 Quantum Computing

Richard P. Feynman asked wondered whether classical computer can simulate a quantum phenomenon[90]. This was the real starting point for a Quantum Computer.

The idea was expounded physically by Paul Benioff[91]. A detailed discussion for a working, physical Quantum Computer will follow in later chapters. The theory for Quantum Computation was laid down by David Deutsch in 1985[92]. He showed that, in principle, a quantum computer is perfectly capable of stimulating quantum phenomena. A working Quantum Computer was dreamed to be a device that computes on non-classical (quantum) theory. It should take into account (and benefit) from the counter-intuitive idea of superposition, entanglement, parallelism in Quantum Mechanics to manipulate data. There has been a major interest in Quantum Computing because of its advanced benefits in civilian and security purposes by using the principles of Quantum Mechanics[93][94]. Contrasted with a classical computer, this machine also has its variant of the Turing model, as developed by David Deutsch. The rigorous details for the theory of quantum computation are not of concern in this thesis; what matters for now is that a model and its machinery have been rigorously defined[70] and that this machinery can be seen to be applied directly.

3.1 Fundamentals

The fundamental unit of information and for computation on a Quantum Computer is the quantum bit or qubit for short, deriving its analogy on its counterpart in classical computing[70]. A bit has two possible states, 0 and 1. Correspondingly, qubits have corresponding states of $|0\rangle$ and $|1\rangle$, called the computational basis, which are orthonormal by definition. What gives quantum computing its power is that a qubit can exist in a superposition $\alpha|0\rangle + \beta|1\rangle$ where $|\alpha|^2 + |\beta|^2 = 1$ until it is measured It is clear that if n-bits of information are needed, we have 2^n possible combinations, classically. In a complete analogy, such possible combinations can be made by $(|0\rangle + |1\rangle)^{\otimes n}$ with an appropriate normalisation. Thus, a two qubit system is of the form $\alpha_{00}|00\rangle + \alpha_{10}|10\rangle + \alpha_{01}|01\rangle + \alpha_{11}|11\rangle$ for $|\alpha_{00}|^2 + |\alpha_{10}|^2 + |\alpha_{01}|^2 + |\alpha_{11}|^2 = 1$

Since α and β both vary over a continuous range, this superposition state can, in principle, contain infinite amount of information for different values of α and β in just one qubit! This is, however, misleading since a measurement will yield either $|0\rangle$ or $|1\rangle$ with probabilities $|\alpha|^2$ and $|\beta|^2$, respectively. Thus a single measurement yields only a single bit of information. In order to determine the values of α and β, an infinite number of such states have to be measured. Of course, this is experimentally not feasible. If only one bit can be measured at a time, then how is a Quantum Computer powerful? The answer lies in the fact that manipulations, before a measurement, on qubits can be performed as though there were more than one quantum bits of information because the state exists in a superposition. This is called parallelism.

To manipulate information, unitary transformations are carried out. Classically, this is done in chips. This manipulated data is sent along a board in wires. Correspondingly, a quantum circuit has wires and elementary quantum gates. The algorithms that run on a Quantum Computer can be pictorially represented using this scheme. These wires, however, do not necessarily indicate the movement of a qubit, it also has the flexibility to correspond to the passage of time is read from left to right. For an example, see the Deutsch-Joza algorithm described in the next section.

There are a few elementary quantum gates that are universal in a sense that any one-bit quantum gate can be built from such gates[95]. To look at a few examples of quantum gates, consider first the simple NOT gate. Classically, a NOT gate changes 0 to 1 and vice versa. The Quantum NOT gate acts similarly on a single qubit $\alpha|0\rangle + \beta|1\rangle$ and changes it to $\beta|0\rangle + \alpha|1\rangle$. This is represented by the Projection Operator $\hat{X} = |1\rangle\langle 0| + |0\rangle\langle 1|$. This is equal to the Pauli operator $\sigma_{10} + \sigma_{01}$. A gate, called the $\hat{Z}$ gate, that acts on a qubit $\alpha|0\rangle + \beta|1\rangle$ to yield $\alpha|0\rangle - \beta|1\rangle$ corresponds to σ_z with appropriate basis. Another important gate is the Hadamard Gate $\hat{H} := \frac{|0\rangle\langle 0|+|1\rangle\langle 0|+|1\rangle\langle 0|+|1\rangle\langle 1|}{\sqrt{2}}$, which has been used extensively in this thesis. The only restriction on defining gates is to have unitary transformations because they preserve the topology of the vectors and are invertible. The transformations defined above are unitary.

Proof. $\hat{X}^\dagger\hat{X} = (|1\rangle\langle 0| + |0\rangle\langle 1|)\,(|1\rangle\langle 0| + |0\rangle\langle 1|)$

$= |1\rangle\langle 0|1\rangle\langle 0| + |1\rangle\langle 0|0\rangle\langle 1| + |0\rangle\langle 1|1\rangle\langle 0| + |0\rangle\langle 1|0\rangle\langle 1|$

$= 0 + |1\rangle\langle 1| + |0\rangle\langle 0| + 0 = I$

$\hat{Z}^\dagger\hat{Z} = (|1\rangle\langle 1| + |0\rangle\langle 0|)\,(|1\rangle\langle 1| + |0\rangle\langle 0|)$

$= |1\rangle\langle 1|1\rangle\langle 1| + |1\rangle\langle 1|0\rangle\langle 0| + |0\rangle\langle 0|1\rangle\langle 1| + |0\rangle\langle 0|0\rangle\langle 0| = I$

and $\hat{H}^\dagger\hat{H} = \frac{|0\rangle\langle 0|+|1\rangle\langle 0|+|1\rangle\langle 0|+|1\rangle\langle 1|}{\sqrt{2}}\,\frac{|0\rangle\langle 0|+|1\rangle\langle 0|+|1\rangle\langle 0|+|1\rangle\langle 1|}{\sqrt{2}}$

$$= \frac{1}{2}\begin{pmatrix} |0\rangle\langle 0|0\rangle\langle 0| + |0\rangle\langle 0|1\rangle\langle 0| + |0\rangle\langle 0|1\rangle\langle 0| + |0\rangle\langle 0|1\rangle\langle 1| + \\ |1\rangle\langle 0|0\rangle\langle 0| + |1\rangle\langle 0|1\rangle\langle 0| + |1\rangle\langle 0|1\rangle\langle 0| + |1\rangle\langle 0|1\rangle\langle 1| + \\ |1\rangle\langle 0|0\rangle\langle 0| + |1\rangle\langle 0|1\rangle\langle 0| + |1\rangle\langle 0|1\rangle\langle 0| + |1\rangle\langle 0|1\rangle\langle 1| + \\ |1\rangle\langle 1|0\rangle\langle 0| + |1\rangle\langle 1|1\rangle\langle 0| + |1\rangle\langle 1|1\rangle\langle 0| + |1\rangle\langle 1|1\rangle\langle 1| \end{pmatrix}$$

$= \frac{1}{2}\,(2|1\rangle\langle 1| + 2|0\rangle\langle 0|) = I$ ■

For multiple qubits, the tensor product of operators is used. Each tensor acts on its corresponding vector i.e. $\left(\hat{P}_1 \otimes \hat{P}_2 \otimes ... \otimes \hat{P}_n\right)(|j\rangle_1 \otimes |j\rangle_2 \otimes ... \otimes |j\rangle_n) =$

$\hat{P}_1|j\rangle_1 \otimes \hat{P}_2|j\rangle_2 \otimes ... \otimes \hat{P}_n|j\rangle_n$. The mathematics developed for such tensors has been defined in the appendix section.

3.2 Deutsch-Jozsa Algorithm

The Deutsch Jozsa algorithm is a quantum algorithm that demonstrates the simple and elegant power of a quantum computer. This algorithm proposed by David Deutsch and Richard Jozsa in 1985[96] sparked interest in Quantum Computing because it demonstrated its power and elegance. The algorithm that was initially proposed was not, in fact, deterministic. However, an improvement by Cleveland in 1998[97] improved the algorithm, which is still referred to as the Deutsch-Jozsa algorithm in their honour. The algorithm asks us to suppose that we have an unknown Boolean function. Classically, every value of such an m-tuple function would need to be evaluated to determine the function completely. If the function is promised to be either constant or balanced in its output, then, classically, it would take $2^{m-1}+1$ inputs to determine the function completely. After applying the algorithm, only one evaluation is needed to determine the function completely. The algorithm works as follows[84] :-

1. Take an initial state $|\psi\rangle = |0\rangle^{\otimes m} \otimes |1\rangle$
2. Apply the Hadamard operator on the $m+1$ qubits. This transforms the state $|\psi\rangle$ to $H^{\otimes m+1}|\psi\rangle = |\psi'\rangle$

$$|\psi'\rangle = H^m|0\rangle^m \otimes H|1\rangle = \frac{1}{\sqrt{2^m}}\left(\sum_{x\in\{0,1\}^m}|x\rangle\right)\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

   which is easy to verify

3. Apply a unitary operator $U_f|x,y\rangle = |x\rangle \otimes |y \oplus f(x)\rangle$ on $|\psi'\rangle$ to get

$$|\psi''\rangle = \frac{1}{\sqrt{2^{m+1}}}\sum_{x\in\{0,1\}^m}|x\rangle\otimes|0\oplus f(x)\rangle - \frac{1}{\sqrt{2^{m+1}}}\sum_{x\in\{0,1\}^m}|x\rangle\otimes|1\oplus f(x)\rangle$$

$$|\psi''\rangle = \frac{1}{\sqrt{2^{m+1}}}\sum_{x\in\{0,1\}^m}|x\rangle\otimes|f(x)\rangle - \frac{1}{\sqrt{2^{m+1}}}\sum_{x\in\{0,1\}^m}|x\rangle\otimes|1\oplus f(x)\rangle$$

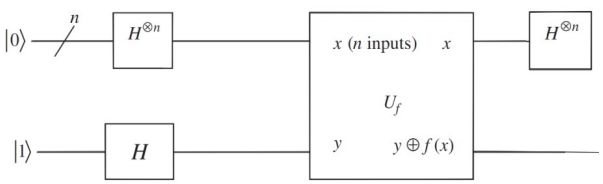
   If $f(x)=0$, then we have the state

$$\frac{1}{\sqrt{2^{m+1}}}\sum_{x\in\{0,1\}^m}|x\rangle\otimes|0\rangle - \frac{1}{\sqrt{2^{m+1}}}\sum_{x\in\{0,1\}^m}|x\rangle\otimes|1\rangle$$

$$= \frac{1}{\sqrt{2^m}}\sum_{x\in\{0,1\}^m}|x\rangle\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2^m}}\sum_{x\in\{0,1\}^m}(-1)^0|x\rangle\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2^m}}\sum_{x\in\{0,1\}^m}(-1)^{f(x)}|x\rangle\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

Diagram



3.jpg

Figure 3: A circuit diagram for Deutsch-Jozsa Algorithm

If $f(x) = 1$, then we have

$|\psi''\rangle = \frac{1}{\sqrt{2^{m+1}}} \sum_{x \in \{0,1\}^m} |x\rangle \otimes |1\rangle - \frac{1}{\sqrt{2^{m+1}}} \sum_{x \in \{0,1\}^m} |x\rangle \otimes |0\rangle$

$\frac{1}{\sqrt{2^m}} \sum_{x \in \{0,1\}^m} |x\rangle \otimes \left(\frac{|1\rangle - |0\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2^m}} \sum_{x \in \{0,1\}^m} (-1)|x\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

$= \frac{1}{\sqrt{2^m}} \sum_{x \in \{0,1\}^m} (-1)^{f(x)}|x\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

i.e. in either case we have

$$|\psi''\rangle = \frac{1}{\sqrt{2^m}} \sum_{x \in \{0,1\}^m} (-1)^{f(x)}|x\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

4. Apply $H^{\otimes m} \otimes I|\psi''\rangle$ to get

$$|\psi_{final}\rangle = \frac{1}{\sqrt{2^m}} \left[\left(\sum_{x \in \{0,1\}^m} (-1)^{f(x)}\right) \left(\frac{1}{\sqrt{2^m}} \sum_{y \in \{0,1\}^m} |y\rangle\right)\right] \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$\text{or } \frac{1}{2^m} \left(\sum_{y \in \{0,1\}^m} \sum_{x \in \{0,1\}^m} (-1)^{\langle x,y \rangle + f(x)}|y\rangle\right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

where $\langle x, y \rangle = x_1 y_1 \oplus x_2 y_2 \oplus ... \oplus x_m y_m$

Light shone, and order from disorder sprung - John Milton

4 Quantum Optics

Newton's corpuscular hypothesis was greatly challenged by the beautiful unification of electricity with magnetism when J. C. Maxwell introduced the following famous equations[20]:-

$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$

$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

$\nabla . \mathbf{B} = 0$

$\nabla . \mathbf{D} = \rho$

where $\mathbf{H}$ is the magnetic vector, $\mathbf{E}$ is the electric vector, $\mathbf{J}$ is the current density, ρ the charge density and $\mathbf{J} = \sigma \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$, $\mathbf{D} = \epsilon \mathbf{E}$. Here, μ and ϵ denote the permittivity and permeability of the medium, respectively and σ is the specific conductivity. ϵ is also called the dielectric constant. From these empirical observations rose the wave equation[9]

$$\nabla^2 \mathbf{D} - \frac{1}{c^2} \frac{\partial^2 \mathbf{D}}{\partial t^2} \text{ for } \mathbf{D} = \mathbf{D}(\mathbf{r}, t).$$

A similar equation is derived for the magnetic vector. The solution of these wave equations[10] are

$$E_x(y,t) = \left(2\nu^2 / V\varepsilon_0\right)^{1/2} q(t) \sin(kz)$$

and

$$B_x(y,t) = \left(2\mu_0^2 \varepsilon_0 \nu^2 / Vk^2\right)^{1/2} \dot{q}(t) \sin(kz)$$

. The propagation of such waves through a medium whose permittivity and permeability varies with space and time may be completely described by such an equation. However, Maxwell's equations may also be used to determine the diffraction of light by ultrasonic waves[12]. This is possible by using appropriate boundary conditions with the wave equation. Thus most diffraction problems are solvable using such a boundary value problem. This approach suffices for many classical problems with the diffracting element conveniently considered as an induced current. The dipole approximation of a (classical) field interacting with an atom is one example. However, such an approach fails in many cases[83]. The first idea would be have a quantised field defined rigorously, through canonical operators. Historically, this is the route adopted by Hiesenberg himself.

4.1 Field Quantisation

Suppose $\mathbf{D} = \mathbf{D}(\mathbf{r}, t)$ is a field inside a cavity resonator of length L will have the wave number $k_j = j\pi/L$ for $j = 1, 2, 3, ...$ and $\upsilon_j = jc\pi/L$[83]. This leads us to consider

$$\mathbf{D}_x(y,t) = \sum A_j q_j \sin(k_j y) \tag{1}$$

[9] see appendix for derivation

[10] see appendix for derivation

for a linearly polarised electric field in the cavity with expanded normal mode amplitudes of the cavity indexed with j. The complementary magnetic field is given by

$$H_z = \sum A_j \frac{\dot{q}_j \varepsilon_0}{k_j} \cos(k_j y) \tag{2}$$

The cavity has a volume $V = LA$ where A is the transverse area of the optical cavity resonator. The axis of propagation is taken as the y-axis. Our purpose is to discretise this field by explicitly expressing independent oscillations. Under such a consideration, it will do well to the reader to imagine a simple harmonic oscillator, since originally, the analogy was constructed in this way. The amplitude of the field is analogous to the amplitude of the vibration of a simple harmonic oscillator. Classically, the Hamiltonian is given by

$$H = \frac{1}{2} \int dV [\varepsilon_0 E_x^2(y,t) + \frac{1}{\mu_0} B_z^2(y,t)]$$

A straight forward substitution of the solution of the magnetic and electric wave equation gives

$$H = \frac{1}{2}(p^2 + \upsilon^2 q^2)$$

Correspondingly, (2) and (1) give

$$H = \frac{1}{2} \sum \left(m_j \upsilon_j^2 q_j^2 + \frac{p_j^2}{m_j} \right)$$

where m_j is the mass of the oscillator. Quantum Mechanically (here is where quantisation takes place), this is replaced by operators that satisfy $[\hat{p}, \hat{q}] = i\hbar$. One such definition takes the form

$$\hat{p} = -\left(\frac{\hat{a}^\dagger - \hat{a}}{2i} \right) \sqrt{2\hbar m \upsilon} \text{ and } \hat{q} = \left(\frac{\hat{a}^\dagger + \hat{a}}{2m\nu} \right) \sqrt{2\hbar m \upsilon}$$

from which we get

$$\hat{a}_j = \frac{1}{\sqrt{2m_j \hbar \upsilon_j}} (m_j \upsilon_j \hat{q}_j + i\hat{p}_j) \text{ and } \hat{a}_j^\dagger = \frac{1}{\sqrt{2m_j \hbar \upsilon_j}} (m_j \upsilon_j \hat{q}_j - i\hat{p}_j)$$

It can be noted that $\left[\hat{a}_j, \hat{a}_k^\dagger\right] = \delta_{jk}$, $\left[\hat{a}_k^\dagger, \hat{a}_j\right] = -\delta_{jk}$. These operators are called the annihilation and creation operators, respectively and are very important. The reason for calling them such will become clear in the next section. For now, it is to be noted that substituting the canonical momentum and position operator definition in the Hamiltonian, we get

$$H = \hbar \sum \upsilon_j \left(\hat{a}_j^\dagger \hat{a}_j + \frac{1}{2} \right)$$

4.2 Fock States

Let H be a Hilbert space for a single particle and S be a symmetry operator. Define $S\left(H^{\otimes 0}\right) = \mathbb{C}$. Then, F is Fock space if $F(H) = \bigoplus_{n=0}^{\infty} S\left(H^{\otimes n}\right)$. An element a Fock space with a well-defined number of particles is known as a Fock state or number state, named after Vladimir Aleksandrovich Fock.

Suppose there exists a field with frequency υ having the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$, respectively. Let $|n\rangle$ be the energy eigenstate corresponding to the energy eigen value E_n[83] i.e.

$$\mathcal{H}|n\rangle = \hbar\upsilon\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|n\rangle = E_n|n\rangle$$

If we apply the operator $\hat{a}$ from the left, we obtain

$$\mathcal{H}\hat{a}|n\rangle = \hbar\upsilon\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hat{a}|n\rangle = \hat{a}E_n|n\rangle$$

Since $\left[\hat{a}, \hat{a}^\dagger\right] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$

$$\mathcal{H}\hat{a}|n\rangle = \hbar\upsilon\left(\hat{a}\hat{a}^\dagger - 1 + \frac{1}{2}\right)\hat{a}|n\rangle$$
$$\mathcal{H}\hat{a}|n\rangle = \left(E_n - \hbar\upsilon\right)\hat{a}|n\rangle$$

The operation of $\hat{a}$ on a Fock state reduces its energy by $\hbar\upsilon$! More rigorously,

$$\mathcal{H}\alpha_n|n-1\rangle = \left(E_n - \hbar\upsilon\right)\alpha_n|n-1\rangle$$
$$\Rightarrow E_n = E_{n-1} + \hbar\upsilon$$

For $n = 1$, we end up with the ground state $E_0 = E_1 - \hbar\upsilon$. Furthermore, $\mathcal{H}\hat{a}|0\rangle = \left(E_0 - \hbar\upsilon\right)\hat{a}|0\rangle$ is not allowed because energies lower than the ground energy are disallowed. Hence, one must conclude that $\hat{a}|0\rangle = 0$. To find the ground energy, we use

$$\mathcal{H}|0\rangle = \hbar\upsilon\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|0\rangle = E_0|0\rangle$$
$$\Rightarrow E_0 = \frac{\hbar\upsilon}{2}$$

From $E_n = E_{n-1} + \hbar\upsilon$, we have $E_1 = E_0 + \hbar\upsilon = \frac{3}{2}\hbar\upsilon$, $E_2 = E_1 + \hbar\upsilon = \frac{5}{2}\hbar\upsilon$. Continuing this way, $E_n = (\frac{1}{2} + n)\hbar\upsilon$. This is the energy of an n-level. Plugging this value as an eigen energy of the Hamiltonian, we see that

$$\mathcal{H}|n\rangle = \hbar\upsilon\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|n\rangle = \left(\frac{1}{2} + n\right)\hbar\upsilon|n\rangle$$
$$= \hbar\upsilon\hat{a}^\dagger\hat{a}|n\rangle + \frac{\hbar\upsilon}{2}|n\rangle = \frac{\hbar\upsilon}{2}|n\rangle + \hbar\upsilon n|n\rangle$$
$$\Rightarrow \hat{a}^\dagger\hat{a} = n$$

From the normalisation condition,

$$1 = \langle n-1|n-1\rangle = \langle n|\hat{a}^\dagger\hat{a}|n\rangle$$
$$1 = n\langle n|n\rangle = |\alpha_n|^2$$
$$\Rightarrow |\alpha_n| = \sqrt{n}$$
$$\text{Hence } \hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

From this, we can proceed to calculate the eigen value of the Hermitian of $\hat{a}$ as follows:

$$\hat{a}^\dagger\hat{a}|n\rangle = \sqrt{n}\hat{a}^\dagger|n-1\rangle$$
$$n|n\rangle = \sqrt{n}\hat{a}^\dagger|n-1\rangle$$
$$\Rightarrow \sqrt{n+1}\hat{a}^\dagger|n\rangle = (n+1)\,|n+1\rangle$$
$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

4.3 Atom-field interaction

When light falls on an object, it is either reflected or refracted. To explain this behaviour requires an enormous amount of calculations and theory. In order to build a basic theory, we start off with an atom interacting with a field. The physical realisation of the Deustch-Jozsa algorithm in this thesis will requires an atom resonant or off-resonant with a driving field, so this theory is important.

4.3.1 Atom-field interaction Hamiltonian

The Hamiltonian of an atom interacting with a field is the sum of the energy of the atom, of the field and of the interaction. Classically, this interaction is represented by a dipole. If the field is classical in nature, one considers the dipole formation as the energy of the atom-field interaction. This is the semi-classical treatment of the problem. For a fully quantum treatment, as is valid in Bragg Diffraction, the field must be quantised. Quantum Mechanically, the total energy of the field in the cavity is the sum of the energy of each individual vibration[83]. We have

$$\mathcal{H} = \mathcal{H}_A + \mathcal{H}_F + \mathcal{H}_I$$

where $\mathcal{H}_A$ is the Hamiltonian of the atom, $\mathcal{H}_F$ is the total energy of the radiation field and $\mathcal{H}_I$ that of the atom-field interaction. If we approximate the interaction to a dipole and consider individual vibrations of the atom and field, then $\mathcal{H}_I = -e\mathbf{r}.\mathbf{E}$ where $\mathbf{r}$ is the position vector of the dipole formed between the atom and the field, $\mathbf{E}$ is an operator of the radiation field itself and e is the elementary charge. $\mathbf{E}$ is considered to be uniform. This is obtained in analogy from $\mathbf{E}(\mathbf{r},t) = \sum_{\mathbf{k}}\overline{\hat{\epsilon}_{\mathbf{k}}\xi_{\mathbf{k}}}\hat{a}^\dagger_{\mathbf{k}}e^{iv_{\mathbf{k}}t-i\mathbf{k}.\mathbf{r}} + \sum_{\mathbf{k}}\hat{\epsilon}_{\mathbf{k}}\xi_{\mathbf{k}}\hat{a}_{\mathbf{k}}e^{-iv_{\mathbf{k}}t+i\mathbf{k}.\mathbf{r}}$ for each mode $\mathbf{k}$. Here, v is the transition frequency of the atom, $\hat{\epsilon}_{\mathbf{k}}$ is the polarisation direction,

$\hat{a}_{\mathbf{k}}^{\dagger}$ and $\hat{a}_{\mathbf{k}}$ are the creation and annihilation operators satisfying $\left[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}^{\dagger}\right] = 1$, $\xi_{\mathbf{k}} = \frac{\hbar \upsilon_{\mathbf{k}}}{2\epsilon_0 V}$ ϵ_0 is the permittivity of free space and V is the volume of the cavity , As established, $\mathcal{H}_F = \sum_{\mathbf{k}} \hbar \nu_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}}^{\dagger} \hat{a}_{\mathbf{k}} + \frac{1}{2}\right)$. Let $\sum_i |i\rangle\langle i|$ represent a complete set of atomic energy eigenstates i.e. $\sum_i |i\rangle\langle i| = 1$. Since $\mathcal{H}_A|i\rangle = E_i|i\rangle$then

$$\mathcal{H}_A|i\rangle\langle i| = E_i|i\rangle\langle i|$$
$$\sum_i \mathcal{H}_A|i\rangle\langle i| = \sum_i E_i|i\rangle\langle i|$$
$$\mathcal{H}_A = \sum_i E_i \sigma_{ii}$$

*In the last line, the atomic transition (Pauli) operator $\sigma_{nm} = |n\rangle\langle m|$ has been used. Furthermore,

$$e\mathbf{r} = \sum_i \sigma_{ii} e\mathbf{r} \sum_j \sigma_{jj}$$
$$e\mathbf{r} = \sum_{i,j} \wp_{ij} \sigma_{ij}$$

where $\wp_{ij} = e\langle i|\mathbf{r}|j\rangle$ is the electric-dipole transition matrix element. The operator $\mathbf{E}$ takes the form

$$\mathbf{E} = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}} \xi_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}^{\dagger}\right)$$

For simplicity's sake, $\hat{\epsilon}_{\mathbf{k}}, \xi_{\mathbf{k}} \in \mathbb{R}\ \forall \mathbf{k}$. Our Hamiltonian now takes the form

$$\mathcal{H} = \sum_i E_i \sigma_{ii} + \sum_{\mathbf{k}} \hbar \nu_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}}^{\dagger} \hat{a}_{\mathbf{k}} + \frac{1}{2}\right) + \hbar \sum_{i,j} \sum_{\mathbf{k}} g_{\mathbf{k}}^{ij} \sigma_{ij} \left(\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}^{\dagger}\right)$$

The substitution $g_{\mathbf{k}}^{ij} = -\frac{\wp_{ij} \hat{\epsilon}_{\mathbf{k}} \xi_{\mathbf{k}}}{\hbar}$ has been made to easen calculations out. Now, considering our case of a two level atom with an excited state $|a\rangle$ and a ground state $|b\rangle$, $\sum_i |i\rangle\langle i| = |a\rangle\langle a| + |b\rangle\langle b| = 1$. Then,

$$\sum_i E_i \sigma_{ii} = E_a|a\rangle\langle a| + E_b|b\rangle\langle b|$$
$$= \frac{1}{2}E_a\sigma_{aa} + \frac{1}{2}E_a\sigma_{aa} + \frac{1}{2}E_b\sigma_{bb} + \frac{1}{2}E_b\sigma_{bb}$$
$$= \frac{1}{2}E_a\sigma_{aa} + \frac{1}{2}E_a\sigma_{aa} + \frac{1}{2}E_b\sigma_{bb} + \frac{1}{2}E_b\sigma_{bb} + \frac{1}{2}E_a\sigma_{bb} - \frac{1}{2}E_a\sigma_{bb} + \frac{1}{2}E_b\sigma_{aa} - \frac{1}{2}E_b\sigma_{aa}$$
$$\frac{1}{2}\hbar\omega\sigma_z + \frac{1}{2}\left(E_a + E_b\right)$$

Furthermore,

$$\begin{aligned}\hbar\sum_{i,j}\sum_{\mathbf{k}} g_{\mathbf{k}}^{ij}\sigma_{ij}\left(\hat{a}_{\mathbf{k}}+\hat{a}_{\mathbf{k}}^{\dagger}\right) &= \hbar\sum_{\mathbf{k}} g_{\mathbf{k}}^{ij}\left(|a\rangle\langle b|+|b\rangle\langle a|\right)\left(\hat{a}_{\mathbf{k}}+\hat{a}_{\mathbf{k}}^{\dagger}\right)\\ &= \hbar\sum_{\mathbf{k}} g_{\mathbf{k}}^{ij}\left(\sigma_{ab}+\sigma_{ba}\right)\left(\hat{a}_{\mathbf{k}}+\hat{a}_{\mathbf{k}}^{\dagger}\right)\end{aligned}$$

The Hamiltonian takes the form

$$\mathcal{H}=\frac{1}{2}\hbar\omega\sigma_z+\frac{1}{2}\left(E_a+E_b\right)+\sum_{\mathbf{k}}\hbar\nu_{\mathbf{k}}\left(\hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}}+\frac{1}{2}\right)+\hbar\sum_{\mathbf{k}} g_{\mathbf{k}}^{ij}\left(\sigma_{ab}+\sigma_{ba}\right)\left(\hat{a}_{\mathbf{k}}+\hat{a}_{\mathbf{k}}^{\dagger}\right)$$

Every Hamiltonian for a light interacting with an atom corresponds to such a Hamiltonian with a few minor changes.

4.3.2 Interaction of a single two-level atom with a single mode field

There are various ways to obtain the state for an atom-field interaction at any instant later in time of which the probability amplitude method, the Heisenberg operator method, gauge invariance method and the unitary time-evolution operator method are popular. Only the first method will be discussed in the detail.

Probability amplitude method The interaction of a quantised atom with a classical field can be explained. This is at times called the semi-classical theory. There are, however, instances where a classical account of the field fails to suffice[83]. This is true, for example, in spontaneous emission in an atomic system. A simplified variant of the above Hamiltonian is given by

$$\mathcal{H}=\hbar g\left(\sigma_{ab}\hat{a}e^{i\Delta t}+\sigma_{ba}\hat{a}^{\dagger}e^{-i\Delta t}\right)$$

where $\Delta=\omega-\nu$. The interaction of an atom in free space can be described by modes of the universe. As of now, we can consider the simplest case of a two level atom interacting with a single-mode field. There are two parts of the Hamiltonian $\mathcal{H}$ viz. the unperturbed Hamiltonian $\mathcal{H}_0$ and a perturbed Hamiltonian $\mathcal{H}_1$. The two levels of the atom $|a\rangle$ and $|b\rangle$ are the eigenstates of the Hamiltonian $\mathcal{H}$. The eigenvalues of each state are $\hbar\omega_a$ and $\hbar\omega_b$ respectively, where $\hbar$ is Planck's constant. For a field having n photons, the wave function of the two level atom can be written as

$$|\psi(t)\rangle=\sum_n C_{a,n}(t)|a,n\rangle+C_{b,n}(t)|b,n\rangle$$

Using Schrodinger's equation, we get a coupled set of differential equations

$$\dot{C}_{a,n}=-ig\sqrt{n+1}e^{i\Delta t}C_{b,n+1}$$

and

$$\dot{C}_{b,n+1}=-ig\sqrt{n+1}e^{-i\Delta t}C_{a,n}$$

which can be solved with the help of the Laplace transform to get, for $\Omega_n^2 = \Delta^2 + 4g^2(n+1)$

$$C_{a,n}(t) = e^{i\Delta t/2}\left\{\begin{array}{c} C_{a,n}(0)\left[\cos\left(\frac{\Omega_n t}{2}\right) - \frac{i\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right] \\ -\frac{2ig\sqrt{n+1}}{\Omega_n}C_{b,n+1}(0)\sin\left(\frac{\Omega_n t}{2}\right)\end{array}\right\}$$

and

$$C_{b,n+1}(t) = e^{-i\Delta t/2}\left\{\begin{array}{c} C_{b,n+1}(0)\left[\cos\left(\frac{\Omega_n t}{2}\right) + \frac{i\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right)\right] \\ -\frac{2ig\sqrt{n+1}}{\Omega_n}C_{a,n}(0)\sin\left(\frac{\Omega_n t}{2}\right)\end{array}\right\}$$

The state vector and the evolution of the atom interacting with the field is completely determined for any t.

4.4 Bragg Diffraction

In their original paper[78], Arbab Ali Khan and M. Sohail Zubairy developed a method for measuring Quantum non-demolition of a Fock state. The technique involves passing atomic waves in a field cavity and then reading out their diffractions to determine the state of the cavity. In the scheme, single two-level atoms with well-defined momentum states pass through an optical cavity. The diffraction is caused by a kick given by the field to the longitudinal component of the momentum. This kick could have been a complete transition of an atomic state with a release of a photon back in the cavity but the detuning Δ between the atomic transition frequency ω and the frequency of the cavity υ is made sufficiently large to avoid such a complete transition. The transverse momentum is treated classically. Upon entering the cavity, the atom may also undergo an even integral of oscillations called Rabi cycles by taking an integer multiple of the momentum of a photon $\hbar k$, where k is the wave number of the cavity field. The state of the atom remains the same when it exits from the cavity. The magnitude of the momentum does not change - only its direction does. The Fock number of the cavity remains the same, too. A crucial point to consider is that there is no information of the cavity in the atom's external states. It is treated combined with the probability amplitudes of the momentum states. There is no need to consider a summation over the number of photons of the cavity since, experimentally, only one atom is fired in the cavity at one time. Thus, the state vector for such an interaction is

$$e^{-i\frac{P_0}{2mh}t}\sum_{l=-\infty}^{\infty} C_{P_l}^{a,n-1}(t)|a,n-1\rangle + C_{P_l}^{b,n}(t)|b,n\rangle$$

where P_0 is the initial momentum of the atom,the states $|a\rangle$ and $|b\rangle$ are the excited and ground states of the atom injected in the cavity, $|n\rangle$ is the Fock state of the cavity In the expression, the state of the atom is shortly represented as the tensor product for the states of the atom and the photon. After l interactions, the momentum of the atom becomes $P_l = P_0 + l\hbar k$. The exponential is

introduced for simplicity's sake. The Hamiltonian for Bragg's regime is

$$\mathcal{H} = \frac{P_x^2}{2M_1} + \frac{\hbar\Delta_1}{2}\sigma_z + \hbar g_1 \cos(kx_1)\left(\sigma_+\hat{a} + \sigma_-\hat{a}^\dagger\right)$$

where P_x is the momentum of atom 1 along the x-axis after leaving the cavity, $M_1$is the mass of atom 1, Δ_1 is the detuning as a difference of the frequency ω_{ab} (between level $|a\rangle$, the excited state, and $|b\rangle$, the ground state, if we assume that the atom has two states, only) and the frequency ω of the field in the cavity. If the initial momentum P_0 of atom 1 is $\frac{l_0}{2}\hbar k_{x_1}$, then $P_x = (l + \frac{l_0}{2})\hbar k_{x_1}$ where l is the order of diffraction and lo is the initial diffraction order. The operators $\sigma_+ = |a\rangle\langle b|$ and $\sigma_- = |b\rangle\langle a|$ are the atomic spin operators for raising and lowering, respectively. $\hat{a}$ and $\hat{a}^\dagger$are the creation and annihilation operators, respectively. $\sigma_z = |a\rangle\langle a| - |b\rangle\langle b|$ is the Pauli inversion operator. Using Schrodinger's equation, we end up with some coupled differential equations that can be solved exactly.

At the end of the paper, this technique has been proposed to be applied to manipulate quantum information. Affirmatively, this approach has proved fruitful in the area of quantum computing via quantum optics as many researchers use this technique to engineer Quantum algorithms. Also, entangled atoms have been produced with the help of Bragg diffraction[98]

Machines take me by surprise with great frequency - Alan Turing

5 Physical Realisation of a Quantum Computer

As mentioned in the introductory paragraph, a physical, working, quantum computing device is inevitable. Such a machine has to be realised in nature. Before embarking on such a mission, it is important to outline what requirements a physical quantum computer must satisfy. Qubits must be input, processed and give an output. To account for the processing of a quantum computer, one must first prioritise qubits to have a robust representation of quantum information in the form of qubits. This choice is important because their quantum computation is all about manipulating qubits. The manipulations performed on the qubits must cover a family of unitary transformations to account for quantum gates. This is done by a controlled Hamiltonian evolving the state. If individual qubits must be transformed, there must exist an equivalent physical way of doing so. Which important unitary transformations should be realised is a question that is of little concern to us in this thesis; all that matters is that unitary gates must have a physical realisation! The states must be prepared easily and may be measurable. After all, a calculator that can take no numbers and give no answer must be immediately disposed of! The qubits must be easily reproduced for as long as necessary. To be able to measure a qubit, one may think of measurement as a conversion from the quantum world to the classical with a corresponding representation. This is "collapsing the wave function" and where projective measurements come into play. The difficulties associated with measurements depend on what sort of system is used.

This raises a question: can they stay isolated from the environment, so that the they do not decohere, or lose themselves to their surrounding by allowing quantum noise to step in, corrupting a desired manipulation? They must not, however, be completely isolated; they must stay accessible so they can be created, manipulated and measured with ease. These considerations are of fundamental importance and are the basic requirements for a physical realisation of a quantum computer. Quantum Mechanical effects are most prominent and easy to study in spins, charges and polarisations. A spinning coin may have been a good candidate, provided that it stay spinning for a long period. Besides, there are no interference effects that can be taken up. A nuclear spin stands as an antagonist to a spinning coin and is employed in an NMR quantum computer but has a drawback when it comes to measurements because their coupling to an external magnetic field is very small. Also, the energy difference between their spin energy and other sources of energy, say kinetic energy is very small, making it difficult to observe spin energy, let alone manipulate them. However, charged atoms may be trapped in a magnetic field and cooled to reduce their kinetic energy, revealing their spin energies and give exquisite control. Then, these ions are used as bits and their selective transitions by monochromatic light. This is the essence of the ion trap quantum computer. Apart from using atoms, photons stand as good candidates for a quantum computer because

photons do not interact easily with each other and most matter. They can be guided along optical fibres without any change in their polarisation. They can be manipulated easily using phaseshifters and beamsplitters. Microwave cavities and optical cavities are strong candidates for an optical quantum computer.

The following table lists a few candidates for a physical realisation of a quantum computer to give some perspective of a wide range of possibilities[70]

System	T_d	T_o	λ
Nuclear Spin	$10^{-2} - 10^{8}$	$10^{-3} - 10^{-6}$	$10^{5} - 10^{14}$
Electron Spin	10^{-3}	10^{-7}	10^{4}
Ion Trap	10^{-1}	10^{-14}	10^{13}
Microwave cavity	10^{0}	10^{-4}	10^{4}
Optical Cavity	10^{-5}	10^{-14}	10^{9}

where T_d is the decoherence time, T_d is the operation time and λ is maximum number of operations. The decoherence time is important because one must know how long a system can remain in its quantum superposition state. The maximum number of operations has been defined as decoherence time divided by operation time.

5.1 Quantum Computing via Quantum Optics

A branch of Quantum Optics known as Cavity Qauntum Electrodynamics (QED) studies the interaction of single atoms to a few optical, monochromatic modes in a cavity to reduce any quantum noise. The single atoms are also kept isolated to keep them from decohering. The state of each photon in the cavity can, in principle, be transferred to and from single atoms. By applying phase shift, phase gates can be engineered[99][100]. Unitary transformations are applied by controlled interaction timings. Depending on the cavity, different Hamiltonians may be applied. For our purpose, it suffices to consider only a two-levelled atom. Quantum Information can be represented by photon states and the states of the atom or both, simultaneously. Different interaction pictures can be used to account for such a model of computation. This includes the Jaynes-Cummings model, Rayliegh diffraction and Bragg's diffraction. The last model has been used for a physical realisation of a quantum computer. The mathematics for this model has been explained in detail in the previous section.

We now turn to the main concern of the thesis.

6 Engineering Deutsch Jozsa Algorithm using techniques of Cavity Quantum Electrodynamics

An atom which can be labelled as atom is fired into the first cavity, which can be called cavity 1. There are only two states of atom 1 viz. $|a\rangle$, the excited state and $|b\rangle$, the ground state. Cavity 1 initially has a photon state of $|n\rangle$, where $n \in \mathbb{N}$. The atom is far detuned from the cavity hence the interaction of atom 1 with cavity 1 is not resonant. A second atom, atom 2, has to be passed through a different cavity in the same configuration as cavity 1 i.e. non-resonantly. A different cavity is needed because the selection of the interaction time is different to correspond to the Deutsch-Jozsa algorithm. After the atoms leave their respective cavities, a superposition of atomic internal states results.

A third cavity, cavity 3, is placed right the first two cavities. Here, the detuning is again made large enough so that the atoms have a non-resonant interaction. This is our U_f. For this, we consider different case of the initial configuration of the atom. If the cavity is initially in a state of $\frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$, where $n = 0, 1$ are photon numbers, a different result is obtained whereas for a configuration of just $|0\rangle$ or $|1\rangle$ gives a different answer. There is a problem, however. Initially, if the cavity is in a superposition state, the atom and the cavity field get entangled. To "free" the atoms from the cavity, a third resonant atom initially in its ground state is passed through this cavity. This atom is then passed through a Ramsey field to release the information of the atoms it has carried with it. This engineers the desired unitary transformation U_f. It must be noted that cavity 3 has to be arranged in a manner similar to an arrangement of f. Finally, the third atom is passed again through a cavity, cavity 4, with the same configuration as cavity 1. The atoms are then measured. For a generalisation, m such atoms will have to be passed through cavity 1, cavity 3 and cavity 4.

6.1 Step 1

We start with the first cavity. An atom, atom 1, is fired in cavity 1. Cavity 1 and atom 1, set-up to use only external degrees of the atom as quantum bits for computation, lead us to consider the off-resonant Hamiltonian.

$$\mathcal{H} = \frac{P_x^2}{2M_1} + \frac{\hbar\Delta_1}{2}\sigma_z + \hbar g_1 \cos(kx_1)(\sigma_+ \hat{a} + \sigma_- \hat{a}^\dagger)$$

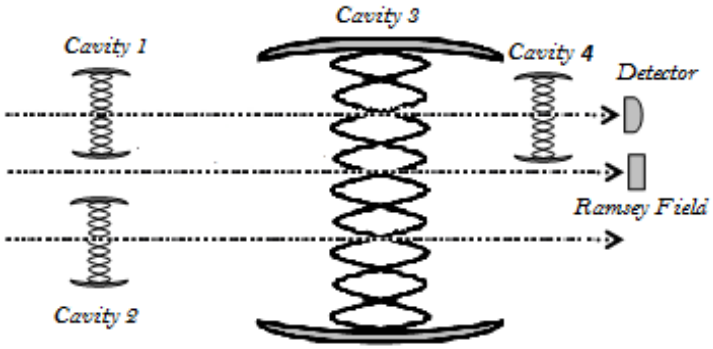
The cavity initially has a photon state of $|n\rangle$. Thus we can suppose a generalised state of

$$|\Psi_{a_1}^{c_1}(t)\rangle = e^{-i\frac{P_0^2}{2M_1}t/\hbar} \sum_{l=-\infty}^{\infty} \left[C_{P_l,a_1,c_1}^{a,n-1}(t)|a, n-1, P_l\rangle + C_{P_l,a_1,c_1}^{b,n}(t)|b, n, P_l\rangle \right] \quad (3)$$

Using the probability amplitude method, we have

$$\frac{\partial}{\partial t}|\Psi_{a_1}^{c_1}(t)\rangle = \frac{-i}{\hbar}\mathcal{H}|\Psi_{a_1}^{c_1}(t)\rangle$$

scheme



4.png

Figure 4: Scheme for implementing the Deutsch-Jozsa Algorithm

$$\frac{\partial}{\partial t}\left(e^{-i\frac{P_0^2}{2M_1}t/\hbar}\right)\sum_{l=-\infty}^{\infty}\left(C^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle+C^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle\right)+$$
$$e^{-i\frac{P_0^2}{2M_1}t/\hbar}\frac{\partial}{\partial t}\left[\sum_{l=-\infty}^{\infty}\left(C^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle+C^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle\right)\right]=$$
$$\frac{-i}{\hbar}\left[\frac{P_x^2}{2M_1}+\frac{\hbar\Delta_1}{2}\sigma_z+\hbar g_1 Cos(kx_1)(\sigma_+\hat{a}+\sigma_-\hat{a}^\dagger)\right]|\Psi^{c_1}_{a_1}(t)\rangle$$

Using $\cos(kx_1)=\frac{e^{ikx_1}+e^{-ikx_1}}{2}$, we get

$$-i\frac{P_0^2}{2M_1\hbar}e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left(C^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle+C^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle\right)+$$
$$e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\dot{C}^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle+e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\dot{C}^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle=$$
$$\frac{-i}{\hbar}\frac{P_{x_1}^2}{2M_1}|\Psi^{c_1}_{a_1}(t)\rangle+\frac{\hbar\Delta_1}{2}|a\rangle\langle a|\Psi^{c_1}_{a_1}(t)\rangle$$
$$\frac{i}{\hbar}\frac{\hbar\Delta_1}{2}|b\rangle\langle b|\Psi^{c_1}_{a_1}(t)\rangle+\frac{-i}{\hbar}\frac{\hbar g_1}{2}e^{ik_{x_1}x}|a\rangle\langle b|\hat{a}|\Psi^{c_1}_{a_1}(t)\rangle+$$
$$\frac{-i}{\hbar}\frac{\hbar g_1}{2}e^{ik_{x_1}x}|b\rangle\langle a|\hat{a}^\dagger|\Psi^{c_1}_{a_1}(t)\rangle+\frac{-i}{\hbar}\frac{\hbar g_1}{2}e^{-ik_{x_1}x}|a\rangle\langle b|\hat{a}|\Psi^{c_1}_{a_1}(t)\rangle$$
$$-\frac{i}{\hbar}\frac{\hbar g_1}{2}e^{-ik_{x_1}x}|b\rangle\langle a|\hat{a}^\dagger|\Psi^{c_1}_{a_1}(t)\rangle \tag{4}$$

It has already been established that $\hat{a}^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle$, $\hat{a}|n\rangle=\sqrt{n}|n-1\rangle$, $\langle b|a\rangle=\langle a|b\rangle=0$, $\langle a|a\rangle=1$, $\langle b|b\rangle=1$, $e^{ikx}|P_l\rangle=|P_{l+1}\rangle$ and $e^{-ikx}|P_l\rangle=|P_{l-1}\rangle$.

Now,

$$|a\rangle\langle a|\Psi_{a_1}^{c_1}(t)\rangle =$$

$$e^{-i\frac{P_0^2}{2M_1}t/\hbar}|a\rangle\langle a|\sum_{l=-\infty}^{\infty}\left(C_{P_l,a_1,c_1}^{a,n-1}(t)|a,n-1,P_l\rangle + C_{P_l,a_1,c_1}^{b,n}(t)|b,n,P_l\rangle\right)$$

$$= e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}|a\rangle\langle a|C_{P_l,a_1,c_1}^{a,n-1}(t)|a,n-1,P_l\rangle + \sum_{l=-\infty}^{\infty}|a\rangle\langle a|C_{P_l,a_1,c_1}^{b,n}(t)|b,n,P_l\rangle$$

$$|a\rangle\langle a|\Psi_{a_1}^{c_1}(t)\rangle = e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}C_{P_l,a_1,c_1}^{a,n-1}(t)|a,n-1,P_l\rangle \tag{5}$$

$$|b\rangle\langle b|\Psi_{a_1}^{c_1}(t)\rangle =$$

$$e^{-i\frac{P_0^2}{2M_1}t/\hbar}|b\rangle\langle b|\sum_{l=-\infty}^{\infty}\left(C_{P_l,a_1,c_1}^{a,n-1}(t)|a,n-1,P_l\rangle + C_{P_l,a_1,c_1}^{b,n}(t)|b,n,P_l\rangle\right)$$

$$= e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}|b\rangle\langle b|C_{P_l,a_1,c_1}^{a,n-1}(t)|a,n-1,P_l\rangle +$$

$$e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}|b\rangle\langle b|C_{P_l,a_1,c_1}^{b,n}(t)|b,n,P_l\rangle$$

$$\text{Hence } |b\rangle\langle b|\Psi_{a_1}^{c_1}(t)\rangle = e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}|C_{P_l,a_1,c_1}^{b,n}(t)|b,n,P_l\rangle \tag{6}$$

$$|a\rangle\langle b|\hat{a}|\Psi_{a_1}^{c_1}(t)\rangle =$$

$$|a\rangle\langle b|\hat{a}e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left(C_{P_l,a_1,c_1}^{a,n-1}(t)|a,n-1,P_l\rangle + C_{P_l,a_1,c_1}^{b,n}(t)|b,n,P_l\rangle\right)$$

$$= e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}C_{P_l,a_1,c_1}^{a,n-1}(t)|a\rangle\langle b|\hat{a}|a,n-1,P_l\rangle +$$

$$e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}C_{P_l,a_1,c_1}^{b,n}(t)|a\rangle\langle b|\hat{a}|b,n,P_l\rangle$$

$$= e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}C_{P_l,a_1,c_1}^{a,n-1}(t)|a\rangle\langle b|\sqrt{n-1}|a,n-2,P_l\rangle +$$

$$e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}C_{P_l,a_1,c_1}^{b,n}(t)|a\rangle\langle b|\sqrt{n}|b,n-1,P_l\rangle$$

$$\text{or } |a\rangle\langle b|\hat{a}|\Psi_{a_1}^{c_1}(t)\rangle = e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sqrt{n}\sum_{l=-\infty}^{\infty}C_{P_l,a_1,c_1}^{b,n}(t)|a,n-1,P_l\rangle$$

$$\text{Hence } e^{ik_x x}|a\rangle\langle b|\hat{a}|\Psi_{a_1}^{c_1}(t)\rangle = \sqrt{n}e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}C_{P_{l+1},a_1,c_1}^{b,n}(t)|a,n-1,P_{l+1}\rangle \tag{7}$$

$$|b\rangle\langle a|\hat{a}^\dagger|\Psi_{a_1}^{c_1}(t)\rangle$$

$$= |b\rangle\langle a|\hat{a}^\dagger e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left(C_{P_l,a_1,c_1}^{a,n-1}(t)|a,n-1,P_l\rangle + C_{P_l,a_1,c_1}^{b,n}(t)|b,n,P_l\rangle\right)$$

$$= e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left(C_{P_l,a_1,c_1}^{a,n-1}(t)|b\rangle\langle a|\hat{a}^\dagger|a,n-1,P_l\rangle + C_{P_l,a_1,c_1}^{b,n}(t)|b\rangle\langle a|\hat{a}^\dagger|b,n,P_l\rangle\right)$$

$$= e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}C_{P_l,a_1,c_1}^{a,n-1}(t)|b\rangle\langle a|\sqrt{n}|a,n,P_l\rangle +$$

$$e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}C_{P_l,a_1,c_1}^{b,n}(t)|b\rangle\langle a|\sqrt{n+1}|b,n+1,P_l\rangle$$

$$= \sqrt{n}e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}C_{P_l,a_1,c_1}^{a,n-1}(t)|b,n,P_l\rangle$$

$$\Rightarrow\ e^{ik_{x_1}x}|b\rangle\langle a|\hat{a}^\dagger|\Psi_{a_1}^{c_1}(t)\rangle = \sqrt{n}e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}C_{P_{l+1},a_1,c_1}^{a,n-1}(t)|b,n,P_{l+1}\rangle \tag{8}$$

$$|a\rangle\langle b|\hat{a}|\Psi_{a_1}^{c_1}(t)\rangle =$$

$$|a\rangle\langle b|\hat{a}e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left(C_{P_l,a_1,c_1}^{a,n-1}(t)|a,n-1,P_l\rangle + C_{P_l,a_1,c_1}^{b,n}(t)|b,n,P_l\rangle\right)$$

$$= \sum_{l=-\infty}^{\infty} C_{P_l,a_1,c_1}^{a,n-1}(t)|a\rangle\langle b|\hat{a}|a,n-1,P_l\rangle + \sum_{l=-\infty}^{\infty} C_{P_l,a_1,c_1}^{b,n}(t)|a\rangle\langle b|\hat{a}|b,n,P_l\rangle$$

$$= \sum_{l=-\infty}^{\infty} C_{P_l,a_1,c_1}^{a,n-1}(t)|a\rangle\langle b|\sqrt{n-1}|a,n-2,P_l\rangle + \sum_{l=-\infty}^{\infty} C_{P_l,a_1,c_1}^{b,n}(t)|a\rangle\langle b|\sqrt{n}|b,n-1,P_l\rangle$$

$$= 0 + \sum_{l=-\infty}^{\infty} C_{P_l,a_1,c_1}^{b,n}(t)\sqrt{n}|a,n-1,P_l\rangle$$

$$\Rightarrow e^{-ik_x x}|a\rangle\langle b|\hat{a}|\Psi_{a_1}^{c_1}(t)\rangle = e^{-ik_{x_1}x}\left[0 + \sum_{l=-\infty}^{\infty} C_{P_l,a_1,c_1}^{b,n}(t)\sqrt{n}|a,n-1,P_l\rangle\right]$$

$$\text{or } e^{-ik_x x}|a\rangle\langle b|\hat{a}|\Psi_{a_1}^{c_1}(t)\rangle = \sqrt{n}\sum_{l=-\infty}^{\infty} C_{P_{l-1},a_1,c_1}^{b,n}(t)|a,n-1,P_{l-1}\rangle \qquad (9)$$

$$|b\rangle\langle a|\hat{a}^\dagger|\Psi_{a_1}^{c_1}(t)\rangle =$$

$$|b\rangle\langle a|\hat{a}^\dagger e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left(C_{P_l,a_1,c_1}^{a,n-1}(t)|a,n-1,P_l\rangle + C_{P_l,a_1,c_1}^{b,n}(t)|b,n,P_l\rangle\right)$$

$$= e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty} C_{P_l,a_1,c_1}^{a,n-1}(t)|b\rangle\langle a|\hat{a}^\dagger|a,n-1,P_l\rangle +$$

$$e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty} C_{P_l,a_1,c_1}^{b,n}(t)|b\rangle\langle a|\hat{a}^\dagger|b,n,P_l\rangle$$

$$= e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty} C_{P_l,a_1,c_1}^{a,n-1}(t)|b\rangle\langle a|\sqrt{n}|a,n,P_l\rangle +$$

$$e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty} C_{P_l,a_1,c_1}^{b,n}(t)|b\rangle\langle a|\sqrt{n+1}|b,n+1,P_l\rangle$$

$$= e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty} C_{P_l,a_1,c_1}^{a,n-1}(t)\sqrt{n}|b,n,P_l\rangle + 0$$

$$\Rightarrow e^{-ik_{x_1}x}|b\rangle\langle a|\hat{a}^\dagger|\Psi_{a_1}^{c_1}(t)\rangle = \sqrt{n}e^{-i\frac{P_{01}^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty} C_{P_{l-1},a_1,c_1}^{a,n-1}(t)|b,n,P_{l-1}\rangle \quad (10)$$

Using Eq. (5), Eq. (6), Eq. (7), Eq. (8), Eq. (9) and Eq. (10) in Eq. (4) we

get

$$-i\frac{P_0^2}{2M_1\hbar}e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left(C^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle+C^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle\right)+$$
$$e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left(\dot{C}^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle+\dot{C}^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle\right)=$$
$$\frac{-i}{\hbar}\frac{P_x^2}{2M_1}e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left(C^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle+C^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle\right)+$$
$$\frac{-i}{\hbar}\frac{\hbar\Delta_1}{2}e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left(C^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle-C^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle\right)+$$
$$\frac{-i}{\hbar}\frac{\hbar g_1}{2}\sqrt{n}e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left(C^{b,n}_{P_{l+1},a_1,c_1}(t)|a,n-1,P_{l+1}\rangle+C^{a,n-1}_{P_{l+1},a_1,c_1}(t)|b,n,P_{l+1}\rangle\right)+$$
$$\frac{-i}{\hbar}\frac{\hbar g_1}{2}\sqrt{n}e^{-i\frac{P_0^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left(C^{a,n-1}_{P_{l-1},a_1,c_1}(t)|b,n,P_{l-1}\rangle+C^{b,n}_{P_{l-1},a_1,c_1}(t)|a,n-1,P_{l-1}\rangle\right)$$

$e^{-i\frac{P_{01}^2}{2M_1}t/\hbar}\neq 0$, therefore

$$-i\frac{P_0^2}{2M_1\hbar}\sum_{l=-\infty}^{\infty}\left(C^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle+C^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle\right)+$$
$$\sum_{l=-\infty}^{\infty}\left(\dot{C}^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle+\dot{C}^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle\right)=$$
$$\frac{-i}{\hbar}\frac{P_x^2}{2M_1}\sum_{l=-\infty}^{\infty}\left(C^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle+C^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle\right)+$$
$$\frac{-i}{\hbar}\frac{\hbar\Delta_1}{2}\sum_{l=-\infty}^{\infty}\left(C^{a,n}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle-C^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle\right)+$$
$$-\frac{i}{\hbar}\frac{\hbar g_1}{2}\sqrt{n}\sum_{l=-\infty}^{\infty}\left(C^{b,n}_{P_{l+1},a_1,c_1}(t)|a,n-1,P_{l+1}\rangle+C^{a,n-1}_{P_{l+1},a_1,c_1}(t)|b,n,P_{l+1}\rangle\right)$$
$$-\frac{i}{\hbar}\frac{\hbar g_1}{2}\sqrt{n}\sum_{l=-\infty}^{\infty}\left(C^{a,n-1}_{P_{l-1},a_1,c_1}(t)|b,n,P_{l-1}\rangle+C^{b,n}_{P_{l-1},a_1,c_1}(t)|a,n-1,P_{l-1}\rangle\right)$$

$$\sum_{l=-\infty}^{\infty} \dot{C}^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle + \sum_{l=-\infty}^{\infty} \dot{C}^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle =$$
$$\frac{-i}{\hbar}\sum_{l=-\infty}^{\infty}\left(\frac{P_x^2}{2M_1}-\frac{P_0^2}{2M_1}+\frac{\hbar\Delta_1}{2}\right)C^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle +$$
$$\frac{-i}{\hbar}\sum_{l=-\infty}^{\infty}\left(\frac{P_x^2}{2M_1}-\frac{P_0^2}{2M_1}-\frac{\hbar\Delta_1}{2}\right)C^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle$$
$$+\frac{-i}{\hbar}\frac{\hbar g_1}{2}\sqrt{n}\sum_{l=-\infty}^{\infty}C^{b,n}_{P_{l+1},a_1,c_1}(t)|a,n-1,P_{l+1}\rangle +$$
$$\frac{-i}{\hbar}\frac{\hbar g_1}{2}\sqrt{n}\sum_{l=-\infty}^{\infty}C^{a,n-1}_{P_{l+1},a_1,c_1}(t)|b,n,P_{l+1}\rangle - \frac{i}{\hbar}\frac{\hbar g_1}{2}\sqrt{n}\sum_{l=-\infty}^{\infty}C^{a,n-1}_{P_{l-1},a_1,c_1}(t)|b,n,P_{l-1}\rangle +$$
$$-\frac{i}{\hbar}\frac{\hbar g_1}{2}\sqrt{n}\sum_{l=-\infty}^{\infty}C^{b,n}_{P_{l-1},a_1,c_1}(t)|a,n-1,P_{l-1}\rangle$$

$P_0^2 = \frac{l_o^2}{4}\hbar^2 k_{x_1}^2$, $P_x^2 = \left(l+\frac{l_o}{2}\right)^2\hbar^2 k_{x_1}^2$, then

$$i\sum_{l=-\infty}^{\infty}\dot{C}^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle + i\sum_{l=-\infty}^{\infty}\dot{C}^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle =$$
$$\sum_{l=-\infty}^{\infty}\left(\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1}+\frac{\Delta_1}{2}\right)C^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle$$
$$+\sum_{l=-\infty}^{\infty}\left(\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1}-\frac{\Delta_1}{2}\right)C^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle$$
$$+\frac{g_1\sqrt{n}}{2}\sum_{l=-\infty}^{\infty}C^{b,n}_{P_{l+1},a_1,c_1}(t)|a,n-1,P_{l+1}\rangle + \frac{g_1\sqrt{n}}{2}\sum_{l=-\infty}^{\infty}C^{a,n-1}_{P_{l+1},a_1,c_1}(t)|b,n,P_{l+1}\rangle +$$
$$\frac{g_1\sqrt{n}}{2}\sum_{l=-\infty}^{\infty}C^{a,n-1}_{P_{l-1},a_1,c_1}(t)|b,n,P_{l-1}\rangle + \frac{g_1\sqrt{n}}{2}\sum_{l=-\infty}^{\infty}C^{b,n}_{P_{l-1},a_1,c_1}(t)|a,n-1,P_{l-1}\rangle$$

Adjusting the summation, taking projection over the available bases and considering the accumulative nature of momentum, we end up with

$$i\dot{C}^{a,n-1}_{P_l,a_1,c_1}(t) = \left[\frac{l(l+l_0)\hbar k_{x_1}^2}{M_1}+\frac{\Delta_1}{2}\right]C^{a,n-1}_{P_l,a_1,c_1}(t) + \frac{g_1\sqrt{n}}{2}\left[C^{b,n}_{P_{l+1},a_1,c_1}(t)+C^{b,n}_{P_{l-1},a_1,c_1}(t)\right] \quad (11)$$

$$\text{and } i\dot{C}^{b,n}_{P_l,a_1,c_1}(t) = \left[\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1}-\frac{\Delta_1}{2}\right]C^{b,n}_{P_l,a_1,c_1}(t) + \frac{g_1\sqrt{n}}{2}\left[C^{a,n-1}_{P_{l-1},a_1,c_1}(t)+C^{a,n-1}_{P_{l+1},a_1,c_1}(t)\right] \quad (12)$$

In the Bragg regime, the recoil energy $\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1}$ much greater than the energy associated with the Rabi frequency $\frac{g\sqrt{n}}{2}$. The slow varying amplitudes will thus dominate the solution to these coupled equations. If we want to separate the slow and the fast moving terms, we need to make the substitution

$$\begin{aligned} C^{a,n-1}_{P_l,a_1,c_1}(t) &= e^{i\Delta_1 t/2} A^{a,n-1}_{P_l,a_1,c_1}(t) \\ C^{b,n}_{P_l,a_1,c_1}(t) &= e^{i\Delta_1 t/2} A^{b,n}_{P_l,a_1,c_1}(t) \end{aligned}$$

Then, Eqs. (11) and (12) take the form

$$\begin{aligned} i\dot{A}^{a,n-1}_{P_l,a_1,c_1}(t) &= \left[\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1} + \Delta_1\right] A^{a,n-1}_{P_l,a_1,c_1}(t) + \frac{g_1\sqrt{n}}{2}\left[A^{b,n}_{P_{l+1},a_1,c_1}(t) + A^{b,n}_{P_{l-1},a_1,c_1}(t)\right] \\ i\dot{A}^{b,n}_{p_l,a_1,c_1}(t) &= \left[\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1}\right] A^{b,n}_{P_l,a_1,c_1}(t) + \frac{g_1\sqrt{n}}{2}\left[A^{a,n-1}_{P_{l-1},a_1,c_1}(t) + A^{a,n-1}_{P_{l+1},a_1,c_1}(t)\right] \end{aligned}$$

In the adiabatic approximation, the recoil energy $\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1}$ is far less than the detuning Δ_1, therefore $\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1} + \Delta_1 \approx \Delta_1$. Then

$$\begin{aligned} i\dot{A}^{a,n-1}_{P_l,a_1,c_1}(t) &= \Delta_1 A^{a,n-1}_{P_l,a_1,c_1}(t) + \frac{g_1\sqrt{n}}{2}\left(A^{b,n}_{P_{l+1},a_1,c_1}(t) + A^{b,n}_{P_{l-1},a_1,c_1}(t)\right) \\ i\dot{A}^{b,n}_{P_l,a_1,c_1}(t) &= \left(\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1}\right) A^{b,n}_{P_l,a_1,c_1}(t) + \frac{g_1\sqrt{n}}{2}\left(A^{a,n-1}_{P_{l-1},a_1,c_1}(t) + A^{a,n-1}_{P_{l+1},a_1,c_1}(t)\right) \end{aligned}$$

The first equation is valid for odd values of l whereas the second equation holds for even values of l. This is because even values of l physically signify a complete cycle from excitation to de-excitation. Effectively, l varies from -3 to 1[**?**]. This means that $A^{b,n}_{P_2,a_1,c_1}(t) = A^{b,n}_{P_{-4},a_1,c_1}(t) = 0 \ \forall \ t$. Furthermore, the law of conservation of energy implies that $\frac{|P_{in}|^2}{2M_1} = \frac{|P_{out}|^2}{2M_1}$

$\implies \frac{\left(l+\frac{l_o}{2}\right)^2\hbar^2 k_{x_1}^2}{2M_1} = \frac{\left(\frac{l_o}{2}\right)^2\hbar^2 k_{x_1}^2}{2M_1}$

$\implies l(l+l_o) = 0$

$\implies l = 0$ or $l = -l_o$

For a first order Bragg diffraction, $l_0 = 2$. Thus the wave function reduces to

$$|\Psi^{c_1}_{a_1}(t)\rangle = C^{b,n}_{P_{-2},a_1,c_1}(t)|b,n,P_{-2}\rangle + C^{b,n}_{P_0,a_1,c_1}(t)|b,n,P_0\rangle$$

This also means that $\dot{A}^{a,n-1}_{P_1,a_1,c_1}(t) = \dot{A}^{a,n-1}_{P_{-3},a_1,c_1}(t) = \dot{A}^{a,n-1}_{P_{-1},a_1,c_1}(t) = 0 \ \forall \ t$. Then, we end up with

$$\begin{aligned} &For\ l = 1 \\ 2i\dot{A}^{a,n-1}_{P_1,a_1,c_1}(t) &= 2\Delta_1 A^{a,n-1}_{P_1,a_1,c_1}(t) + g_1\sqrt{n}A^{b,n}_{P_2,a_1,c_1}(t) + g_1\sqrt{n}A^{b,n}_{P_0,a_1,c_1}(t) \end{aligned}$$

$$\Delta_1 A^{a,n-1}_{P_1,a_1,c_1}(t) = -\frac{g_1\sqrt{n}}{2}A^{b,n}_{P_0,a_1,c_1}(t) \tag{13}$$

$$
\begin{aligned}
For\ l &= -3 \\
2i\dot{A}^{a,n-1}_{P_{-3},a_1,c_1}(t) &= \Delta_1 A^{a,n-1}_{P_{-3},a_1,c_1}(t) + g_1\sqrt{n}A^{b,n}_{P_{-4},a_1,c_1}(t) + g_1\sqrt{n}A^{b,n}_{P_{-2},a_1,c_1}(t)
\end{aligned}
$$

$$
\Delta_1 A^{a,n-1}_{P_{-3},a_1,c_1}(t) = -\frac{g_1\sqrt{n}}{2}A^{b,n}_{P_{-2},a_1,c_1}(t) \tag{14}
$$

$$
\begin{aligned}
For\ l &= -1 \\
2i\dot{A}^{a,n-1}_{P_{-1},a_1,c_1}(t) &= 2\Delta_1 A^{a,n-1}_{P_{-1},a_1,c_1}(t) + g_1\sqrt{n}A^{b,n}_{P_{0},a_1,c_1}(t) + g_1\sqrt{n}A^{b,n}_{P_{-2},a_1,c_1}(t)
\end{aligned}
$$

$$
\Delta_1 A^{a,n-1}_{P_{-1},a_1,c_1}(t) = -\frac{g_1\sqrt{n}}{2}\left[A^{b,n}_{P_{0},a_1,c_1}(t) + A^{b,n}_{P_{-2},a_1,c_1}(t)\right] \tag{15}
$$

$$
\begin{aligned}
For\ l &= 0 \\
\dot{A}^{b,n}_{P_{0},a_1,c_1}(t) &= -i\frac{g_1\sqrt{n}}{2}\left[A^{a,n-1}_{P_{-1},a_1,c_1}(t) + A^{a,n-1}_{P_{1},a_1,c_1}(t)\right] \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
For\ l &= -2 \\
\dot{A}^{b,n}_{P_{-2},a_1,c_1}(t) &= -i\frac{g_1\sqrt{n}}{2}\left[A^{a,n-1}_{P_{-1},a_1,c_1}(t) + A^{a,n-1}_{P_{-3},a_1,c_1}(t)\right] \qquad (17)
\end{aligned}
$$

Substituting Eq. (13) and Eq. (15) in Eq. (16), we get

$$
\dot{A}^{b,n}_{P_{0},a_1,c_1}(t) = \frac{-ig_1\sqrt{n}}{2}\left[\frac{-g_1\sqrt{n}}{2\Delta_1}A^{b,n}_{P_{0},a_1,c_1}(t) - \frac{g_1\sqrt{n}}{2\Delta_1}\left(A^{b,n}_{P_{-2},a_1,c_1}(t) + A^{b,n}_{P_{0},a_1,c_1}(t)\right)\right]
$$

which simplifies to

$$
-i\dot{A}^{b,n}_{P_{0},a_1,c_1}(t) = \frac{g_1^2 n}{2\Delta_1}A^{b,n}_{P_{0},a_1,c_1}(t) + \frac{g_1^2 n}{4\Delta_1}A^{b,n}_{P_{-2},a_1,c_1}(t) \tag{18}
$$

Substituting Eq. (14) and Eq. (15) in Eq. (17), we get

$$
\dot{A}^{b,n}_{P_{-2},a_1,c_1}(t) = \frac{-ig_1\sqrt{n}}{2}\left[\frac{g_1\sqrt{n}}{2\Delta_1}\left(A^{b,n}_{P_{0},a_1,c_1}(t) + A^{b,n}_{P_{-2},a_1,c_1}(t)\right) - \frac{g_1\sqrt{n}}{2\Delta_1}A^{b,n}_{P_{-2},a_1,c_1}(t)\right]
$$

which simplifies to

$$
-i\dot{A}^{b,n}_{P_{-2},a_1,c_1}(t) = \frac{g_1^2 n}{2\Delta_1}A^{b,n}_{P_{-2},a_1,c_1}(t) + \frac{g_1^2 n}{4\Delta_1}A^{b,n}_{P_{0},a_1,c_1}(t) \tag{19}
$$

These two coupled equations can be solved as follows: Let $\mathcal{L}[A^{b,n}_{P_{-2},a_1,c_1}(t)] = F^{b,n}_{P_{-2},a_1,c_1}(s)$ and $\mathcal{L}[A^{b,n}_{P_{0},a_1,c_1}(t)] = F^{b,n}_{P_{0},a_1,c_1}(s)$ where $\mathcal{L}$ denotes the Laplace

transform. Then, $\mathcal{L}[\dot{A}^{b,n}_{P_0,a_1,c_1}(t)] = sF^{b,n}_{P_0,a_1,c_1}(s) - A^{b,n}_{P_0,a_1,c_1}(0)$ and $\mathcal{L}[\dot{A}^{b,n}_{P_{-2},a_1,c_1}(t)] = sF^{b,n}_{P_{-2},a_1,c_1}(s) - A^{b,n}_{P_{-2},a_1,c_1}(0)$. Eq. (18) takes the form

$$-isF^{b,n}_{P_0,a_1,c_1}(s) + iA^{b,n}_{P_0,a_1,c_1}(0) = \frac{g_1^2 n}{2\Delta_1}F^{b,n}_{P_0,a_1,c_1}(s) + \frac{g_1^2 n}{4\Delta_1}F^{b,n}_{P_{-2},a_1,c_1}(s)$$

which can be expressed as

$$F^{b,n}_{P_0,a_1,c_1}(s) = \frac{\frac{g_1^2 n}{4\Delta_1}F^{b,n}_{P_{-2},a_1,c_1}(s) - iA^{b,n}_{P_0,a_1,c_1}(0)}{-is - \frac{g_1^2 n}{2\Delta_1}} \tag{20}$$

Eq. (19) takes the form

$$-isF^{b,n}_{P_{-2},a_1,c_1}(s) + iA^{b,n}_{P_{-2},a_1,c_1}(0) = \frac{g_1^2 n}{2\Delta_1}F^{b,n}_{P_{-2},a_1,c_1}(s) + \frac{g_1^2 n}{4\Delta_1}F^{b,n}_{P_0,a_1,c_1}(s)$$

$$\left(-is - \frac{g_1^2 n}{2\Delta_1}\right)F^{b,n}_{P_{-2},a_1,c_1}(s) = \frac{g_1^2 n}{4\Delta_1}F^{b,n}_{P_0,a_1,c_1}(s) - iA^{b,n}_{P_{-2},a_1,c_1}(0)$$

From Eq. (20), this is

$$\left(-is - \frac{g_1^2 n}{2\Delta_1}\right)F^{b,n}_{P_{-2},a_1,c_1}(s) = \frac{g_1^2 n}{4\Delta_1}\frac{\frac{g_1^2 n}{4\Delta_1}F^{b,n}_{P_{-2},a_1,c_1}(s) - iA^{b,n}_{P_0,a_1,c_1}(0)}{-is - \frac{g_1^2 n}{2\Delta_1}} - iA^{b,n}_{P_{-2},a_1,c_1}(0)$$

$$\left(-is - \frac{g_1^2 n}{2\Delta_1}\right)F^{b,n}_{P_{-2},a_1,c_1}(s) = \frac{\left(\frac{g_1^2 n}{4\Delta_1}\right)^2 F^{b,n}_{P_{-2},a_1,c_1}(s)}{-is - \frac{g_1^2 n}{2\Delta_1}} + i\frac{g_1^2 n}{4\Delta_1}\frac{A^{b,n}_{P_0,a_1,c_1}(0)}{is + \frac{g_1^2 n}{2\Delta_1}} - iA^{b,n}_{P_{-2},a_1,c_1}(0)$$

$$-is - \frac{g_1^2 n}{2\Delta_1} - \frac{\left(\frac{g_1^2 n}{4\Delta_1}\right)^2}{-is - \frac{g_1^2 n}{2\Delta_1}}F^{b,n}_{P_{-2},a_1,c_1}(s) = i\frac{g_1^2 n}{4\Delta_1}\frac{A^{b,n}_{P_0,a_1,c_1}(0)}{is + \frac{g_1^2 n}{2\Delta_1}} - iA^{b,n}_{P_{-2},a_1,c_1}(0)$$

$$\frac{(is)^2 + \left(\frac{g_1^2 n}{2\Delta_1}\right)^2 + 2i\frac{g_1^2 sn}{2\Delta_1} - \left(\frac{g_1^2 n}{4\Delta_1}\right)^2}{-is - \frac{g_1^2 n}{2\Delta_1}}F^{b,n}_{P_{-2},a_1,c_1}(s) = \left(i\frac{g_1^2 n}{4\Delta_1}\right)\frac{A^{b,n}_{P_0,a_1,c_1}(0)}{is + \frac{g_1^2 n}{2\Delta_1}} - iA^{b,n}_{P_{-2},a_1,c_1}(0)$$

$$\frac{\left(is + \frac{g_1^2 n}{2\Delta_1}\right)^2 - \left(\frac{g_1^2 n}{4\Delta_1}\right)^2}{-is - \frac{g_1^2 n}{2\Delta_1}}F^{b,n}_{P_{-2},a_1,c_1}(s) = \left(i\frac{g_1^2 n}{4\Delta_1}\right)\frac{A^{b,n}_{P_0,a_1,c_1}(0)}{is + \frac{g_1^2 n}{2\Delta_1}} - iA^{b,n}_{P_{-2},a_1,c_1}(0)$$

$$i^2\left[\frac{\left(s - i\frac{g_1^2 n}{2\Delta_1}\right)^2 + \left(\frac{g_1^2 n}{4\Delta_1}\right)^2}{-is - \frac{g_1^2 n}{2\Delta_1}}\right]F^{b,n}_{P_{-2},a_1,c_1}(s) = \left(i\frac{g_1^2 n}{4\Delta_1}\right)\frac{A^{b,n}_{P_0,a_1,c_1}(0)}{is + \frac{g_1^2 n}{2\Delta_1}} - iA^{b,n}_{P_{-2},a_1,c_1}(0)$$

$$F^{b,n}_{P_{-2},a_1,c_1}(s) = \left(\frac{is + \frac{g_1^2 n}{2\Delta_1}}{\left(-s + i\frac{g_1^2 n}{2\Delta_1}\right)^2 + \left(\frac{g_1^2 n}{4\Delta_1}\right)^2}\right)\left[\left(i\frac{g_1^2 n}{4\Delta_1}\right)\frac{A^{b,n}_{P_0,a_1,c_1}(0)}{is + \frac{g_1^2 n}{2\Delta_1}} - iA^{b,n}_{P_{-2},a_1,c_1}(0)\right]$$

$$F^{b,n}_{P_{-2},a_1,c_1}(s) = i\frac{\left(\frac{g_1^2 n}{4\Delta_1}\right)}{\left(-s+i\frac{g_1^2 n}{2\Delta_1}\right)^2+\left(\frac{g_1^2 n}{4\Delta_1}\right)^2}A^{b,n}_{P_0,a_1,c_1}(0) - i\frac{is+\frac{g_1^2 n}{2\Delta_1}}{\left(-s+i\frac{g_1^2 n}{2\Delta_1}\right)^2+\left(\frac{g_1^2 n}{4\Delta_1}\right)^2}A^{b,n}_{P_{-2},a_1,c_1}(0)$$

$$F^{b,n}_{P_{-2},a_1,c_1}(s) = i\frac{\left(\frac{g_1^2 n}{4\Delta_1}\right)}{\left(-s+i\frac{g_1^2 n}{2\Delta_1}\right)^2+\left(\frac{g_1^2 n}{4\Delta_1}\right)^2}A^{b,n}_{P_0,a_1,c_1}(0) + \frac{s-i\frac{g_1^2 n}{2\Delta_1}}{\left(-s+i\frac{g_1^2 n}{2\Delta_1}\right)^2+\left(\frac{g_1^2 n}{4\Delta_1}\right)^2}A^{b,n}_{P_{-2},a_1,c_1}(0)$$

$$F^{b,n}_{P_{-2},a_1,c_1}(s) = i\frac{\left(\frac{g_1^2 n}{4\Delta_1}\right)}{\left(s-i\frac{g_1^2 n}{2\Delta_1}\right)^2+\left(\frac{g_1^2 n}{4\Delta_1}\right)^2}A^{b,n}_{P_0,a_1,c_1}(0) + \frac{s-i\frac{g_1^2 n}{2\Delta_1}}{\left(s-i\frac{g_1^2 n}{2\Delta_1}\right)^2+\left(\frac{g_1^2 n}{4\Delta_1}\right)^2}A^{b,n}_{P_{-2},a_1,c_1}(0)$$

From $\mathcal{L}\left[e^{-at}Sin(bt)\right] = \frac{b}{(s+a)^2+b^2}$ and $\mathcal{L}\left[e^{-at}Cos(bt)\right] = \frac{s+a}{(s+a)^2+b^2}$ or, equivalently, $e^{-at}Sin(bt) = \mathcal{L}^{-1}\left[\frac{b}{(s+a)^2+b^2}\right]$ and $e^{-at}Cos(bt) = \mathcal{L}^{-1}\left[\frac{s+a}{(s+a)^2+b^2}\right]$

$$\mathcal{L}^{-1}\left[F^{b,n}_{P_{-2},a_1,c_1}(s)\right] =$$

$$\mathcal{L}^{-1}\left[i\frac{\left(\frac{g_1^2 n}{4\Delta_1}\right)}{\left(s-i\frac{g_1^2 n}{2\Delta_1}\right)^2+\left(\frac{g_1^2 n}{4\Delta_1}\right)^2}A^{b,n}_{P_0,a_1,c_1}(0)\right] + \mathcal{L}^{-1}\left[\frac{s-i\frac{g_1^2 n}{2\Delta_1}}{\left(s-i\frac{g_1^2 n}{2\Delta_1}\right)^2+\left(\frac{g_1^2 n}{4\Delta_1}\right)^2}A^{b,n}_{P_{-2},a_1,c_1}(0)\right]$$

$$\Rightarrow A^{b,n}_{P_{-2},a_1,c_1}(t) = e^{i\frac{g_1^2 n}{2\Delta_1}t}\left[A^{b,n}_{P_{-2},a_1,c_1}(0)\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right) + iA^{b,n}_{P_0,a_1,c_1}(0)\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right)\right]$$

From Eq. (19)

$$-i\dot{A}^{b,n}_{P_{-2},a_1,c_1}(t) = \frac{g_1^2 n}{2\Delta_1}A^{b,n}_{P_{-2},a_1,c_1}(t) + \frac{g_1^2 n}{4\Delta_1}A^{b,n}_{P_0,a_1,c_1}(t)$$

$$-i\frac{g_1^2 n}{4\Delta_1}e^{i\frac{g_1^2 n}{2\Delta_1}t}\left[iA^{b,n}_{P_0,a_1,c_1}(0)\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right) - A^{b,n}_{P_{-2},a_1,c_1}(0)\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right)\right] =$$

$$\frac{g_1^2 n}{2\Delta_1}e^{i\frac{g_1^2 n}{2\Delta_1}t}\left[A^{b,n}_{P_{-2},a_1,c_1}(0)\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right) + iA^{b,n}_{P_0,a_1,c_1}(0)\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right)\right] + \frac{g_1^2 n}{4\Delta_1}A^{b,n}_{P_0,a_1,c_1}(t)$$

This simplifies to

$$A^{b,n}_{P_0,a_1,c_1}(t) = e^{i\frac{g_1^2 n}{2\Delta_1}t}\left[A^{b,n}_{P_0,a_1,c_1}(0)\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right) + iA^{b,n}_{P_{-2},a_1,c_1}(0)\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right)\right]$$

Using the reverse substitutions from $C^{b,n}_{P_{-2},a_1,c_1}(t) = e^{i\Delta_1 t/2}A^{b,n}_{P_{-2},a_1,c_1}(t)$ and $C^{b,n}_{P_0,a_1,c_1}(t) = e^{i\Delta_1 t/2}A^{b,n}_{P_0,a_1,c_1}(t)$, we obtain

$$\begin{aligned} C^{b,n}_{P_{-2},a_1,c_1}(t) &= C^{b,n}_{P_{-2},a_1,c_1}(0)\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right) + iC^{b,n}_{P_0,a_1,c_1}(0)\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right) \\ C^{b,n}_{P_0,a_1,c_1}(t) &= C^{b,n}_{P_0,a_1,c_1}(0)\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right) + iC^{b,n}_{P_{-2},a_1,c_1}(0)\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right) \end{aligned}$$

with the phase factors ignored. This is not illegitimate because the modulus of each such exponential is 1. The above solutions represent sinusoidally varying probabilities. The period of these probabilities depends on the number of photons present in the cavity. The state

$$|\Psi^{c_1}_{a_1}(t)\rangle = C^{b,n}_{P_{-2},a_1,c_1}(t)|b,n,P_{-2}\rangle + C^{b,n}_{P_0,a_1,c_1}(t)|b,n,P_0\rangle$$

has been calculated. Initially, the atom has a momentum state of $|P_0\rangle$.Then, $C^{b,n}_{P_0,a_1,c_1}(0) = 1$ and $C^{b,n}_{P_{-2},a_1,c_1}(0) = 0$ For interaction time $t = \frac{4\Delta_1}{g_1^2 n}\pi/4$, the atom that emerges from cavity 1 is in the state

$$|\Psi^{c_1}_{a_1}(t)\rangle = \frac{|b,n,P_0\rangle + i|b,n,P_{-2}\rangle}{\sqrt{2}}$$

$$|\Psi^{c_1}_{a_1}(t)\rangle = |b,n\rangle \otimes \left(\frac{|P_0\rangle + i|P_{-2}\rangle}{\sqrt{2}}\right)$$

It can be noted that the atom and the cavity are not entangled. This can be shortly written as

$$|\Psi_{a_1}(t)\rangle = \frac{|P_0\rangle + i|P_{-2}\rangle}{\sqrt{2}}$$

This is the Hadamard transform of the bit $|0\rangle$. m such atoms will have to be passed through this cavity with the same interaction time to achieve correspondence with the Deutsch-Jozsa algorithm. Now, a similar 2nd atom is passed through cavity 2. It is again assumed that cavity 2 has a Fock state of $|n\rangle$. That is, we have the state function

$$|\Psi^{c_2}_{a_2}(t)\rangle = e^{-i\frac{P_{02}^2}{2M_2}t/\hbar}\sum_{l=-\infty}^{\infty}\left(C^{a,n-1}_{P_l,a_2,c_2}(t)|a,n-1,P_l\rangle + C^{b,n}_{P_l,a_2,c_2}(t)|b,n,P_l\rangle\right)$$

with Hamiltonian

$$\mathcal{H} = \frac{P_{x_2}^2}{2M_2} + \frac{\hbar\Delta_2}{2}\sigma_z + \hbar g_2 Cos\left(kx_2\right)\left(\sigma_+\hat{a} + \sigma_-\hat{a}^\dagger\right)$$

Going through the familiar series of steps, we end up with

$$|\Psi_{a_2}^{c_2}(t)\rangle = C_{P_{-2},a_2,c_2}^{b,n}(t)|b,n,P_{-2}\rangle + C_{P_0,a_2,c_2}^{b,n}(t)|b,n,P_0\rangle$$

where

$$\begin{aligned} C_{P_{-2},a_2,c_2}^{b,n}(t) &= e^{i\left(\frac{\Delta_2}{2}+\frac{g_2^2 n}{2\Delta_2}\right)t}\left[C_{P_{-2},a_2,c_2}^{b,n}(0)\cos\left(\frac{g_2^2 n}{4\Delta_2}t\right) + iC_{P_0,a_2,c_2}^{b,n}(0)\sin\left(\frac{g_2^2 n}{4\Delta_2}t\right)\right] \\ C_{P_0,a_2,c_2}^{b,n}(t) &= e^{i\left(\frac{\Delta_2}{2}+\frac{g_2^2 n}{2\Delta_2}\right)t}\left[C_{P_0,a_2,c_2}^{b,n}(0)\cos\left(\frac{g_2^2 n}{4\Delta_2}t\right) + iC_{P_{-2},a_2,c_2}^{b,n}(0)\sin\left(\frac{g_2^2 n}{4\Delta_2}t\right)\right] \end{aligned}$$

If, initially, atom 2 has a momentum state of $|P_{-2}\rangle$, then, $C_{P_0,a_1,c_1}^{b,n}(0) = 0$ and $C_{P_{-2},a_1,c_1}^{b,n}(0) = 1$. For an interaction time $t = \frac{4\Delta_2}{g_2^2 n}3\pi/4$, the atom that emerges from cavity 1 is in the state

$$|\Psi_{a_2}(t)\rangle = \frac{i|P_0\rangle - |P_{-2}\rangle}{\sqrt{2}}$$

Except for the unit imaginary, we have achieved an analogy with the Deutsch-Algorithm.

6.2 Step 2

A third cavity is setup similar to cavity 1 and 2. This step will correspond to the unitary transformation U_f. Since the outcome of the unitary transformation is dependent on the values of the function, four different cases will have to be treated. This line of reasoning will become clear as the calculations proceed. In the first two cases, the cavity is in the superposition of $\frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$. In the third and fourth case, the cavity is in the Fock state $|1\rangle$ and $|0\rangle$, respectively. First, two atoms, on after the other, are made to interact off-resonantly in cavity 3. The interaction time for the first atom is $\frac{g_1^2}{4\Delta_1}t = \pi/2$ and $\frac{g_2^2}{4\Delta_2}t = \pi/2$ for the second atom. The atoms and the cavity are entangled for the first two cases. Next, another atom is passed through the same cavity. This time, the detuning of the atom is made as small as possible to have a resonant interaction. For an interaction time $g_3 t = \pi/2$, the cavity is switched off. This atom will then lose its information by being passed through a Ramsey field and then taking projection over the ground state. This results in the disentangling of the two atoms. The second atom has a common phase. There is a difference in the phase of atom 1, as required.

6.2.1 Case I

We send these two atoms to cavity 3, one after the other. If, initially, cavity 3 is in the superposition state $\frac{|1\rangle+|0\rangle}{\sqrt{2}}$ and atom 1 is, as before, in its ground state, the state for atom 1 in cavity 3 $|\Psi_{a_1}^{c_3}(t)\rangle$ will be of the form

$$e^{-i\frac{P_{02}^2}{2M_2}t/\hbar}\sum_{l=-\infty}^{\infty}\left(C_{P_l,a_1,c_3}^{a,0}(t)|a,0,P_l\rangle + C_{P_l,a_1,c_3}^{b,1}(t)|b,1,P_l\rangle + C_{P_l,a_1,c_3}^{b,0}(t)|b,0,P_l\rangle\right)$$

For a cavity set up in likeness of cavity 1 and 2, we use the same Hamiltonian as before. Following a similar series of steps, we get the following equations:

$$i\dot{C}^{a,0}_{P_l,a_1,c_3}(t) = \left(\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1} + \frac{\Delta_1}{2}\right) C^{a,0}_{P_l,a_1,c_3}(t) + \frac{g_1}{2}\left(C^{b,1}_{P_{l+1},a_1,c_3}(t) + C^{b,1}_{P_{l-1},a_1,c_3}(t)\right) \quad (21)$$

$$i\dot{C}^{b,1}_{P_l,a_1,c_3}(t) = \left(\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1} - \frac{\Delta_1}{2}\right) C^{b,1}_{P_l,a_1,c_3}(t) + \frac{g_1}{2}\left(C^{a,0}_{P_{l+1},a_1,c_3}(t) + C^{a,0}_{P_{l-1},a_1,c_3}(t)\right) \quad (22)$$

$$i\dot{C}^{b,0}_{P_l,a_1,c_3}(t) = \left(\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1} - \frac{\Delta_1}{2}\right) C^{b,0}_{P_l,a_1,c_3}(t) \quad (23)$$

$$i\dot{C}^{a,0}_{P_l,a_1,c_3}(t) = \left(\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1} + \frac{\Delta_1}{2}\right) C^{a,0}_{P_l,a_1,c_3}(t) \quad (24)$$

From Eq. (23) can be solved to yield

$$\implies C^{b,0}_{P_l,a_1,c_3}(t) = C^{b,0}_{P_l,a_1,c_3}(0) e^{-i\left(\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1} - \frac{\Delta_1}{2}\right)t}$$

This equation is valid for even values of $l \in \{-1, 0, 1, 2, 3\}$, i.e. for $l = 0, -l_0 = -2$. For a first order Bragg diffraction, $l_0 = 2$. Thus, for such values, $C^{b,0}_{P_l,a_1,c_3}(t)$ is constant. These values for l are obtained from the law of conservation of energy by using the same reasoning that was done previously. Similarly, from 24

$$i\dot{C}^{a,0}_{P_l,a_1,c_3}(t) = \left(\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1} + \frac{\Delta_1}{2}\right) C^{a,0}_{P_l,a_1,c_3}(t)$$

$$\implies C^{a,0}_{P_l,a_1,c_3}(t) = C^{a,0}_{P_l,a_1,c_3}(0) e^{-i\left(\frac{l(l+l_0)\hbar k_{x_1}^2}{2M_1} + \frac{\Delta_1}{2}\right)t}$$

Equations (21) and (22) are coupled. Again, we separate the fast and slow moving terms by the substitution

$$C^{a,0}_{P_l,a_1,c_3}(t) = e^{i\Delta_1 t/2} A^{a,0}_{P_l,a_1,c_3}(t)$$

and

$$C^{b,1}_{P_l,a_1,c_3}(t) = e^{i\Delta_1 t/2} A^{b,1}_{P_l,a_1,c_3}(t)$$

and get the same coupled equations as before, for $n = 1$. The solution to these coupled equations is thus the same as above for $n = 1$. Then, the state of the atom after exiting the cavity is

$$\begin{aligned}|\Psi^{c_3}_{a_1}(t)\rangle \quad = \quad & C^{b,1}_{P_0,a_1,c_3}(t)|b,1,P_0\rangle + C^{b,1}_{P_{-2},a_1,c_3}(t)|b,1,P_{-2}\rangle + C^{b,0}_{P_0,a_1,c_3}(t)|b,0,P_0\rangle + \\ & C^{b,0}_{P_{-2},a_1,c_3}(t)|b,0,P_{-2}\rangle + C^{a,1}_{P_{-2},a_1,c_3}(t)|a,1,P_{-2}\rangle + C^{a,1}_{P_0,a_1,c_3}(t)|a,1,P_0\rangle\end{aligned}$$

where

$$C^{b,1}_{P_0,a_1,c_3}(t) = C^{b,1}_{P_0,a_1,c_3}(0)\cos\left(\frac{g_1^2}{4\Delta_1}t\right) + iC^{b,1}_{P_{-2},a_1,c_3}(0)\sin\left(\frac{g_1^2}{4\Delta_1}t\right)$$

$$C^{b,1}_{P_{-2},a_1,c_3}(t) = C^{b,1}_{P_{-2},a_1,c_3}(0)\cos\left(\frac{g_1^2}{4\Delta_1}t\right) + iC^{b,1}_{P_0,a_1,c_3}(0)\sin\left(\frac{g_1^2}{4\Delta_1}t\right)$$

$$C^{b,0}_{P_0,a_1,c_3}(t) = C^{b,0}_{P_0,a_1,c_3}(0)$$
$$C^{b,0}_{P_{-2},a_1,c_3}(t) = C^{b,0}_{P_{-2},a_1,c_3}(0)$$
$$C^{a,0}_{P_0,a_1,c_3}(t) = C^{a,0}_{P_0,a_1,c_3}(0)$$
$$C^{a,1}_{P_0,a_1,c_3}(t) = C^{a,1}_{P_0,a_1,c_3}(0)$$

The phase factors have been ignored for simplicity's sake. It can be noted that $\left(\frac{|1\rangle+|0\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|P_0\rangle+i|P_{-2}\rangle}{\sqrt{2}}\right) = \frac{1}{2}\left(|1,P_0\rangle + |0,P_0\rangle + i|1,P_{-2}\rangle + i|0,P_{-2}\rangle\right)$ implies that $C^{b,1}_{P_{-2},a_1,c_3}(0) = C^{b,0}_{P_{-2},a_1,c_3} = i/2$ whereas $C^{b,0}_{P_0,a_1,c_3}(0) = C^{b,1}_{P_0,a_1,c_3}(0) = 1/2$ where $\frac{|1\rangle+|0\rangle}{\sqrt{2}}$ is the initial state of the cavity and $\frac{|P_0\rangle+i|P_{-2}\rangle}{\sqrt{2}}$ is the initial state of the atom. Furthermore, $C^{a,1}_{P_0,a_1,c_3}(0) = C^{a,1}_{P_{-2},a_1,c_3}(0) = 0$. Then, the state of the atom after emerging from the cavity takes the form, For an interaction time $\frac{g_1^2}{4\Delta_1}t = \pi/2$

$$|\Psi^{c_3}_{a_1}(t)\rangle = \frac{1}{2}\left[|0,P_0\rangle - |1,P_0\rangle + i|0,P_{-2}\rangle + i|1,P_{-2}\rangle\right]$$

This state can not be simplified. Thus, cavity 3 and the external states of the atom are entangled. Now, atom 2 is passed through the same cavity, cavity 3. Again, we use an off-resonant interaction. The mathematics is again achieved by the same Hamiltonian acting on the state

$$
\begin{aligned}
|\Psi^{c_3}_{a_1,a_2}(t)\rangle = e^{-i\frac{P_{02}^2}{2M_2}t/\hbar}\sum_{l=-\infty}^{\infty} C^{a,0,P_0^{(1)}}_{P_l^{(2)},a_1,a_2,c_3}(t)|a,0,P_0^{(1)},P_l^{(2)}\rangle + \\
e^{-i\frac{P_{02}^2}{2M_2}t/\hbar}\sum_{l=-\infty}^{\infty} C^{b,1,P_0^{(1)}}_{P_l^{(2)},a_1,a_2,c_3}(t)|b,1,P_0^{(1)},P_l^{(2)}\rangle + \\
e^{-i\frac{P_{02}^2}{2M_2}t/\hbar}\sum_{l=-\infty}^{\infty} C^{b,0,P_0^{(1)}}_{P_l^{(2)},a_1,a_2,c_3}(t)|b,0,P_0^{(1)},P_l^{(2)}\rangle + \\
e^{-i\frac{P_{02}^2}{2M_2}t/\hbar}\sum_{l=-\infty}^{\infty} C^{a,0,P_{-2}^{(1)}}_{P_l^{(2)},a_1,a_2,c_3}(t)|a,0,P_{-2}^{(1)},P_l^{(2)}\rangle + \\
e^{-i\frac{P_{02}^2}{2M_2}t/\hbar}\sum_{l=-\infty}^{\infty} C^{b,1,P_{-2}^{(1)}}_{P_l^{(2)},a_1,a_2,c_3}(t)|b,1,P_{-2}^{(1)},P_l^{(2)}\rangle + \\
e^{-i\frac{P_{02}^2}{2M_2}t/\hbar}\sum_{l=-\infty}^{\infty} C^{b,0,P_{-2}^{(1)}}_{P_l^{(2)},a_1,a_2,c_3}(t)|b,0,P_{-2}^{(1)},P_l^{(2)}\rangle
\end{aligned}
$$

where$P_j^{(1)}$ is the momentum state of the first atom for $j=0,-2$ and $P_l^{(2)}$ is the momentum state of the second atom for doubly infinite values of l. By the probability amplitude method, we get the following equations

$$
\begin{aligned}
i\dot{C}^{a,0,P_j^{(1)}}_{P_l^{(2)},a_1,a_2,c_3}(t) &= \left(\frac{l\left(l+l_0\right)\hbar k_{x_1}^2}{2M_1}+\frac{\Delta_2}{2}\right)C^{a,0,P_j^{(1)}}_{P_l^{(2)},a_1,a_2,c_3}(t) + \\
&\quad \frac{g_2}{2}\left(C^{b,1,P_j^{(1)}}_{P_{l-1}^{(2)},a_2,c_2}(t)+C^{b,1,P_j^{(1)}}_{P_{l+1}^{(2)},a_1,a_2,c_3}(t)\right) \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
i\dot{C}^{b,1,P_j^{(1)}}_{P_l^{(2)},a_1,a_2,c_3}(t) &= \left(\frac{l\left(l+l_0\right)\hbar k_{x_1}^2}{2M_1}-\frac{\Delta_2}{2}\right)C^{b,1,P_j^{(1)}}_{P_l^{(2)},a_1,a_2,c_3}(t) + \\
&\quad \frac{g_2}{2}\left(C^{a,0,P_j^{(1)}}_{P_{l+1}^{(2)},a_1,a_2,c_3}(t)+C^{a,0,P_j^{(1)}}_{P_{l-1}^{(2)},a_1,a_2,c_2}(t)\right) \qquad (26)
\end{aligned}
$$

$$
i\dot{C}^{b,0,P_j^{(1)}}_{P_l^{(2)},a_1,a_2,c_3}(t) = \left(\frac{l\left(l+l_0\right)\hbar k_{x_1}^2}{2M_1}-\frac{\Delta_2}{2}\right)C^{b,0,P_j^{(1)}}_{P_l^{(2)},a_1,a_2,c_3}(t) \qquad (27)
$$

After ignoring the phase factors, we get

$$
\begin{aligned}
|\Psi^{c_3}_{a_1,a_2}(t)\rangle = C^{b,1,P_0^{(1)}}_{P_0^{(2)},a_1,a_2,c_3}(0)\cos\left(\frac{g_2^2}{4\Delta_2}t\right)|b,1,P_0^{(1)},P_0^{(2)}\rangle + \\
iC^{b,1,P_0^{(1)}}_{P_{-2}^{(2)},a_1,a_2,c_3}(0)\sin\left(\frac{g_2^2}{4\Delta_2}t\right)|b,1,P_0^{(1)},P_0^{(2)}\rangle + \\
C^{b,1,P_{-2}^{(1)}}_{P_0^{(2)},a_1,a_2,c_3}(0)\cos\left(\frac{g_2^2}{4\Delta_2}t\right)|b,1,P_{-2}^{(1)},P_0^{(2)}\rangle + \\
iC^{b,1,P_{-2}^{(1)}}_{P_{-2}^{(2)},a_1,a_2,c_3}(0)\sin\left(\frac{g_2^2}{4\Delta_2}t\right)|b,1,P_{-2}^{(1)},P_0^{(2)}\rangle \\
+C^{b,1,P_0^{(1)}}_{P_{-2}^{(2)},a_1,a_2,c_3}(0)\cos\left(\frac{g_2^2}{4\Delta_2}t\right)|b,1,P_0^{(1)},P_{-2}^{(2)}\rangle + \\
iC^{b,1,P_0^{(1)}}_{P_0^{(2)},a_1,a_2,c_3}(0)\sin\left(\frac{g_2^2}{4\Delta_2}t\right)|b,1,P_0^{(1)},P_{-2}^{(2)}\rangle \\
+C^{b,1,P_{-2}^{(1)}}_{P_{-2}^{(2)},a_1,a_2,c_3}(0)\cos\left(\frac{g_2^2}{4\Delta_2}t\right)|b,1,P_{-2}^{(1)},P_{-2}^{(2)}\rangle + \\
iC^{b,1,P_{-2}^{(1)}}_{P_0^{(2)},a_1,a_2,c_3}(0)\sin\left(\frac{g_2^2}{4\Delta_2}t\right)|b,1,P_{-2}^{(1)},P_{-2}^{(2)}\rangle + \\
C^{b,0,P_0^{(1)}}_{P_{-2}^{(2)},a_1,a_2,c_3}(0)|b,0,P_0^{(1)},P_{-2}^{(2)}\rangle + C^{b,0,P_0^{(1)}}_{P_0^{(2)},a_1,a_2,c_3}(0)|b,0,P_0^{(1)},P_0^{(2)}\rangle + \\
C^{b,0,P_{-2}^{(1)}}_{P_{-2}^{(2)},a_1,a_2,c_3}(0)|b,0,P_{-2}^{(1)},P_{-2}^{(2)}\rangle + C^{b,0,P_{-2}^{(1)}}_{P_0^{(2)},a_1,a_2,c_3}(0)|b,0,P_{-2}^{(1)},P_0^{(2)}\rangle +
\end{aligned}
$$

We get our initial conditions from $\frac{1}{2}\left[|0,P_0\rangle - |1,P_0\rangle + i|0,P_{-2}\rangle + i|1,P_{-2}\rangle\right] \otimes \left(\frac{i|P_0^{(2)}\rangle - |P_{-2}^{(2)}\rangle}{\sqrt{2}}\right)$. For an interaction time $\frac{g_2^2}{4\Delta_2}t = \pi/2$, the state $|\Psi_{a_1,a_2}(t)\rangle$ of the atom after emerging from the cavity is

$$
\frac{1}{2\sqrt{2}}\left[\begin{array}{c} i|1,P_0^{(1)},P_0^{(2)}\rangle + |1,P_{-2}^{(1)},P_0^{(2)}\rangle + |1,P_0^{(1)},P_{-2}^{(2)}\rangle - i|1,P_{-2}^{(1)},P_{-2}^{(2)}\rangle + \\ i|0,P_0^{(1)},P_0^{(2)}\rangle - |0,P_{-2}^{(1)},P_0^{(2)}\rangle - |0,P_0^{(1)},P_{-2}^{(2)}\rangle - i|0,P_{-2}^{(1)},P_{-2}^{(2)}\rangle \end{array}\right]
$$

This interaction time is the same for both the atoms fired in cavity 3 if $\Delta_2 = \Delta_1$ and if $g_2 = g_1$. The two atoms are entangled with the cavity. To "free" the atoms from the cavity, a third atom with two states $|a\rangle$, the excited state and $|b\rangle$, the ground state, is passed through the same cavity. This time, the detuning Δ_3 is not made large enough so as to utilise the internal states of atom 3. The on-resonant Hamiltonian for atom 3 is

$$
\mathcal{H} = \hbar g_3\left(\sigma_+\hat{a} + \sigma_-\hat{a}^\dagger\right)
$$

If we assume that atom 3 has only two states viz. $|a\rangle$, the excited state, and $|b\rangle$, the ground state, then the state of the atom before entering the cavity can

be taken to be

$$|\Psi^{c_3}_{a_{1,}a_2,a_3}(t)\rangle = \begin{array}{c} C^{b,1,P^{(1)}_0,P^{(2)}_0}_{a_{1,}a_2,c_3}(t)|b,1,P^{(1)}_0,P^{(2)}_0\rangle + C^{b,1,P^{(1)}_0,P^{(2)}_{-2}}_{a_{1,}a_2,c_3}(t)|b,1,P^{(1)}_0,P^{(2)}_{-2}\rangle + \\ C^{b,0,P^{(1)}_0,P^{(2)}_0}_{a_{1,}a_2,c_3}(t)|b,0,P^{(1)}_0,P^{(2)}_0\rangle + C^{b,0,P^{(1)}_0,P^{(2)}_{-2}}_{a_{1,}a_2,c_3}(t)|b,0,P^{(1)}_0,P^{(2)}_{-2}\rangle + \\ C^{b,1,P^{(1)}_{-2},P^{(2)}_0}_{a_{1,}a_2,c_3}(t)|b,1,P^{(1)}_{-2},P^{(2)}_0\rangle + C^{b,1,P^{(1)}_{-2},P^{(2)}_{-2}}_{a_{1,}a_2,c_3}(t)|b,1,P^{(1)}_{-2},P^{(2)}_{-2}\rangle + \\ C^{b,0,P^{(1)}_{-2},P^{(2)}_0}_{a_{1,}a_2,c_3}(t)|b,0,P^{(1)}_{-2},P^{(2)}_0\rangle + C^{b,0,P^{(1)}_{-2},P^{(2)}_{-2}}_{a_{1,}a_2,c_3}(t)|b,0,P^{(1)}_{-2},P^{(2)}_{-2}\rangle + \\ C^{a,0,P^{(1)}_0,P^{(2)}_{-2}}_{a_{1,}a_2,c_3}(t)|a,0,P^{(1)}_0,P^{(2)}_{-2}\rangle + C^{a,0,P^{(1)}_0,P^{(2)}_0}_{a_{1,}a_2,c_3}(t)|a,0,P^{(1)}_0,P^{(2)}_0\rangle + \\ C^{a,0,P^{(1)}_{-2},P^{(2)}_0}_{a_{1,}a_2,c_3}(t)|a,0,P^{(1)}_{-2},P^{(2)}_0\rangle + C^{a,0,P^{(1)}_{-2},P^{(2)}_{-2}}_{a_{1,}a_2,c_3}(t)|a,0,P^{(1)}_{-2},P^{(2)}_{-2}\rangle \end{array}$$

For the time being, the state can be compressed into

$$|\Psi^{c_3}_{a_{1,}a_2,a_3}(t)\rangle = C^{b,1}_{a_{1,}a_2,c_3}(t)|b,1\rangle + C^{b,0}_{a_{1,}a_2,c_3}(t)|b,0\rangle + C^{a,0}_{a_{1,}a_2,c_3}(t)|a,0\rangle$$

By the probability amplitude method,

$$\frac{\partial}{\partial t}|\Psi(t)\rangle = \frac{-i}{\hbar}\mathcal{H}|\Psi(t)\rangle$$

$$\frac{\partial}{\partial t}C^{b,1}_{a_{1,}a_2,c_3}(t)|b,1\rangle + \frac{\partial}{\partial t}C^{b,0}_{a_{1,}a_2,c_3}(t)|b,0\rangle + \frac{\partial}{\partial t}C^{a,0}_{a_{1,}a_2,c_3}(t)|a,0\rangle + \\ -\frac{i}{\hbar}\hbar g_3\left(\sigma_+\hat{a} + \sigma_-\hat{a}^\dagger\right)|\Psi^{c_3}_{a_{1,}a_2,a_3}(t)\rangle$$

$$\dot{C}^{b,1}_{a_{1,}a_2,c_3}(t)|b,1\rangle + \dot{C}^{b,0}_{a_{1,}a_2,c_3}(t)|b,0\rangle + \dot{C}^{a,0}_{a_{1,}a_2,c_3}(t)|a,0\rangle = \\ -ig_3\left(|a\rangle\langle b|\hat{a} + |b\rangle\langle a|\hat{a}^\dagger\right)|\Psi^{c_3}_{a_{1,}a_2,a_3}(t)\rangle$$

$$\dot{C}^{b,1}_{a_{1,}a_2,c_3}(t)|b,1\rangle + \dot{C}^{b,0}_{a_{1,}a_2,c_3}(t)|b,0\rangle + \dot{C}^{a,0}_{a_{1,}a_2,c_3}(t)|a,0\rangle = \\ -ig_3\left(C^{b,1}_{a_{1,}a_2,c_3}(t)|a,0\rangle + C^{a,0}_{a_{1,}a_2,c_3}(t)|b,1\rangle\right)$$

This gives us the following rate equations

$$i\dot{C}^{b,1}_{a_{1,}a_2,c_3}(t) = g_3 C^{a,0}_{a_{1,}a_2,c_3}(t) \tag{28}$$

$$i\dot{C}^{a,0}_{a_{1,}a_2,c_3}(t) = g_3 C^{b,1}_{a_{1,}a_2,c_3} \tag{29}$$

$$i\dot{C}^{b,0}_{a_{1,}a_2,c_3}(t) = 0$$

The last equation yields $C^{b,0}_{a_{1,}a_2,c_3}(t) = C^{b,0}_{a_{1,}a_2,c_3}(0)$. Applying the Laplace transform on the remaining two, we get

$$isF^{b,1}_{a_{1,}a_2,c_3}(s) - iC^{b,1}_{a_{1,}a_2,c_3}(0) = g_3 F^{a,0}_{a_{1,}a_2,c_3}(s) \tag{30}$$

and

$$isF^{a,0}_{a_{1,}a_2,c_3}(s) - iC^{a,0}_{a_{1,}a_2,c_3}(0) = g_3 F^{b,1}_{a_{1,}a_2,c_3}(s)$$

$$\implies F^{a,0}_{a_1,a_2,c_3}(s) = \frac{1}{is}\left[iC^{a,0}_{a_1,a_2,c_3}(0) + g_3 F^{b,1}_{a_1,a_2,c_3}(t)\right] \tag{31}$$

Using (31) in (30),

$$isF^{b,1}_{a_1,a_2,c_3}(s) - iC^{b,1}_{a_1,a_2,c_3}(0) = \frac{g_3}{is}\left[iC^{a,0}_{a_1,a_2,c_3}(0) + g_3 F^{b,1}_{a_1,a_2,c_3}(t)\right]$$

$$-s^2F^{b,1}_{a_1,a_2,c_3}(s) + sC^{b,1}_{a_1,a_2,c_3}(0) = ig_3C^{a,0}_{a_1,a_2,c_3}(0) + g_3^2F^{b,1}_{a_1,a_2,c_3}(t)$$

$$s^2F^{b,1}_{a_1,a_2,c_3}(s) - sC^{b,1}_{a_1,a_2,c_3}(0) = -ig_3C^{a,0}_{a_1,a_2,c_3}(0) - g_3^2F^{b,1}_{a_1,a_2,c_3}(t)$$

$$\left(s^2 + g_3^2\right)F^{b,1}_{a_1,a_2,c_3}(s) = sC^{b,1}_{a_1,a_2,c_3}(0) - ig_3C^{a,0}_{a_1,a_2,c_3}(0)$$

$$F^{b,1}_{a_1,a_2,c_3}(s) = \frac{s}{s^2+g_3^2}C^{b,1}_{a_1,a_2,c_3}(0) - i\frac{g_3}{s^2+g_3^2}C^{a,0}_{a_1,a_2,c_3}(0)$$

Since $\mathcal{L}\left[e^{-at}\sin(bt)\right] = \frac{b}{(s+a)^2+b^2}$ and $\mathcal{L}\left[e^{-at}\cos(bt)\right] = \frac{a}{(s+a)^2+b^2}$, $\therefore$

$$\mathcal{L}^{-1}\left[F^{b,1}_{a_1,a_2,c_3}(s)\right] = \mathcal{L}^{-1}\left[\frac{s}{s^2+g_3^2}\right]C^{b,1}_{a_1,a_2,c_3}(0) - \mathcal{L}^{-1}\left[\frac{g_3}{s^2+g_3^2}\right]C^{a,0}_{a_1,a_2,c_3}(0)$$

$$\Rightarrow C^{b,1}_{a_1,a_2,a_3,c_3}(t) = C^{b,1}_{a_1,a_2,a_3,c_3}(0)\cos(g_3t) - iC^{a,0}_{a_1,a_2,a_3,c_3}(0)\sin(g_3t)$$

Substituting this into Eq. (28), we obtain

$$i\dot{C}^{b,1}_{a_1,a_2,c_3}(t) = g_3C^{a,0}_{a_1,a_2,c_3}(t)$$

$$-ig_3\left[C^{b,1}_{a_1,a_2,a_3,c_3}(0)Sin(g_3t) + iC^{a,0}_{a_1,a_2,a_3,c_3}(0)Cos(g_3t)\right] = g_3C^{a,0}_{a_1,a_2,c_3}(t)$$

$$C^{a,0}_{a_1,a_2,c_3}(t) = C^{a,0}_{a_1,a_2,a_3,c_3}(0)\cos(g_3t) - iC^{b,1}_{a_1,a_2,a_3,c_3}(0)\sin(g_3t)$$

The state vector

$$\begin{aligned}|\Psi^{c_3}_{a_1,a_2,a_3}(t)\rangle = {} & C^{a,0,P_0^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(t)|a,0,P_0^{(1)},P_0^{(2)}\rangle + C^{b,0,P_0^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(t)|b,0,P_0^{(1)},P_0^{(2)}\rangle + \\ & C^{a,0,P_0^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(t)|a,0,P_0^{(1)},P_{-2}^{(2)}\rangle + C^{b,0,P_0^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(t)|b,0,P_0^{(1)},P_{-2}^{(2)}\rangle + \\ & C^{a,0,P_{-2}^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(t)|a,0,P_{-2}^{(1)},P_0^{(2)}\rangle + C^{b,0,P_{-2}^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(t)|b,0,P_{-2}^{(1)},P_0^{(2)}\rangle + \\ & C^{a,0,P_{-2}^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(t)|a,0,P_{-2}^{(1)},P_{-2}^{(2)}\rangle + C^{b,0,P_{-2}^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(t)|b,0,P_{-2}^{(1)},P_{-2}^{(2)}\rangle + \\ & C^{b,1,P_0^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(t)|b,1,P_0^{(1)},P_0^{(2)}\rangle + C^{b,1,P_0^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(t)|b,1,P_0^{(1)},P_{-2}^{(2)}\rangle + \\ & C^{b,1,P_{-2}^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(t)|b,1,P_{-2}^{(1)},P_0^{(2)}\rangle + C^{b,1,P_{-2}^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(t)|b,1,P_{-2}^{(1)},P_{-2}^{(2)}\rangle\end{aligned}$$

becomes

$$|\Psi^{c_3}_{a_1,a_2,a_3}(t)\rangle =$$
$$\left[C^{a,0,P_0^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(0)\cos(g_3t) - iC^{b,1,P_0^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(0)\sin(g_3t)\right]|a,0,P_0^{(1)},P_0^{(2)}\rangle +$$
$$C^{b,0,P_0^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(0)|b,0,P_0^{(1)},P_0^{(2)}\rangle + C^{b,0,P_0^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(0)|b,0,P_0^{(1)},P_{-2}^{(2)}\rangle +$$
$$\left[C^{a,0,P_0^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(0)\cos(g_3t) - iC^{b,1,P_0^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(0)\sin(g_3t)\right]|a,0,P_0^{(1)},P_{-2}^{(2)}\rangle +$$
$$\left[C^{a,0,P_{-2}^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(0)\cos(g_3t) - iC^{b,1,P_{-2}^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(0)\sin(g_3t)\right]|a,0,P_{-2}^{(1)},P_0^{(2)}\rangle +$$
$$C^{b,0,P_{-2}^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(0)|b,0,P_{-2}^{(1)},P_0^{(2)}\rangle + C^{b,0,P_{-2}^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(0)|b,0,P_{-2}^{(1)},P_{-2}^{(2)}\rangle +$$
$$\left[C^{a,0,P_{-2}^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(0)\cos(g_3t) - iC^{b,1,P_{-2}^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(0)\sin(g_3t)\right]|a,0,P_{-2}^{(1)},P_{-2}^{(2)}\rangle +$$
$$\left[C^{b,1,P_0^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(0)\cos(g_3t) - C^{a,0,P_0^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(0)\sin(g_3t)\right]|b,1,P_0^{(1)},P_0^{(2)}\rangle +$$
$$\left[C^{b,1,P_0^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(0)\cos(g_3t) - C^{a,0,P_0^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(0)\sin(g_3t)\right]|b,1,P_0^{(1)},P_{-2}^{(2)}\rangle +$$
$$\left[C^{b,1,P_{-2}^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(0)\cos(g_3t) - C^{a,0,P_{-2}^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(0)\sin(g_3t)\right]|b,1,P_{-2}^{(1)},P_0^{(2)}\rangle +$$
$$\left[C^{b,1,P_{-2}^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(0)\cos(g_3t) - C^{a,0,P_{-2}^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(0)\sin(g_3t)\right]|b,1,P_{-2}^{(1)},P_{-2}^{(2)}\rangle$$

Initially, the atom is in its ground state. Then, $C^{a,0,P_0^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(0) = C^{a,0,P_0^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(0) = C^{a,0,P_{-2}^{(1)},P_0^{(2)}}_{a_1,a_2,a_3,c_3}(0) = C^{a,0,P_{-2}^{(1)},P_{-2}^{(2)}}_{a_1,a_2,a_3,c_3}(0) = 0$.

Plugging these values in the state vector

$$|\Psi^{c_3}_{a_1,a_2,a_3}(t)\rangle =$$
$$\tfrac{1}{2\sqrt{2}}|0,b\rangle \otimes \left[-i|P_0^{(1)},P_0^{(2)}\rangle - |P_0^{(1)},P_{-2}^{(2)}\rangle - |P_{-2}^{(1)},P_0^{(2)}\rangle - i|P_{-2}^{(1)},P_{-2}^{(2)}\rangle\right] +$$
$$\tfrac{1}{2\sqrt{2}}|0,a\rangle \otimes \left[|P_0^{(1)},P_0^{(2)}\rangle - i|P_0^{(1)},P_{-2}^{(2)}\rangle - i|P_{-2}^{(1)},P_0^{(2)}\rangle + |P_{-2}^{(1)},P_{-2}^{(2)}\rangle\right]$$

Atom 3 is passed through a Ramsey field. This changes the state to

$$|\Psi^{c_3}_{a_1,a_2,a_3}(t)\rangle =$$
$$\tfrac{1}{2\sqrt{2}}\tfrac{|0,b\rangle+|0,a\rangle}{\sqrt{2}} \otimes \left[-i|P_0^{(1)},P_0^{(2)}\rangle - |P_0^{(1)},P_{-2}^{(2)}\rangle - |P_{-2}^{(1)},P_0^{(2)}\rangle - i|P_{-2}^{(1)},P_{-2}^{(2)}\rangle\right] +$$
$$\tfrac{1}{2\sqrt{2}}\tfrac{|0,b\rangle-|0,a\rangle}{\sqrt{2}} \otimes \left[|P_0^{(1)},P_0^{(2)}\rangle - i|P_0^{(1)},P_{-2}^{(2)}\rangle - i|P_{-2}^{(1)},P_0^{(2)}\rangle + |P_{-2}^{(1)},P_{-2}^{(2)}\rangle\right]$$

Then, applying $P = |b\rangle\langle b|$ on $|\Psi^{c_3}_{a_1,a_2,a_3}(t)\rangle$ and renormalising, we get

$$|\Psi_{a_1,a_2,a_3}(t)\rangle = \left(\frac{|P_0^{(1)}\rangle - i|P_{-2}^{(1)}\rangle}{\sqrt{2}}\right)\left(\frac{i|P_0^{(2)}\rangle - |P_{-2}^{(2)}\rangle}{\sqrt{2}}\right)$$

Atom 3 is now "disentangled" with atom 1 and 2. This is analogous to $|\Psi_{out}(t)\rangle = \left(\frac{(-1)^{f(0)}|0\rangle+(-1)^{f(1)}|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$

6.2.2 Case II

This is the same as case I, with only a change in signs. Thus we can use our previous calculations. From $\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|P_0\rangle+i|P_{-2}\rangle}{\sqrt{2}}\right) = \frac{1}{2}\left(|0P_0\rangle - |1P_0\rangle - i|1P_{-2}\rangle + i|0P_{-2}\rangle\right)$ we get our initial conditions for the same state. After applying $\frac{g_1^2}{4\Delta_1}t = \pi/2$, we get

$$|\Psi_{a_1}^{c_3}(t)\rangle = \frac{1}{2}|1,P_0\rangle - \frac{i}{2}|1,P_{-2}\rangle + \frac{1}{2}|0,P_0\rangle + \frac{i}{2}|0,P_{-2}\rangle$$

This is atom 1 leaving cavity 3. Atom 2 now enters cavity 3. Using initial conditions from $\frac{1}{2}\left(|1,P_0^{(1)}\rangle - i|1,P_{-2}^{(1)}\rangle + |0,P_0^{(1)}\rangle + i|0,P_{-2}^{(1)}\rangle\right)\otimes\left(\frac{i|P_0^{(2)}\rangle-|P_{-2}^{(2)}\rangle}{\sqrt{2}}\right)$ and choosing the interaction time $\left(g_2^2/4\Delta_2\right)t = \pi/2$, we get

$$|\Psi_{a_1,a_2}^{c_3}(t)\rangle = \frac{1}{2\sqrt{2}}\left[\begin{array}{c} -i|1,P_0^{(1)},P_0^{(2)}\rangle - |1,P_{-2}^{(1)},P_0^{(2)}\rangle - |1,P_0^{(1)},P_{-2}^{(2)}\rangle - |0,P_{-2}^{(1)},P_0^{(2)}\rangle + \\ i|1,P_{-2}^{(1)},P_{-2}^{(2)}\rangle - |0,P_0^{(1)},P_{-2}^{(2)}\rangle + i|0,P_0^{(1)},P_0^{(2)}\rangle - i|0,P_{-2}^{(1)},P_{-2}^{(2)}\rangle \end{array}\right]$$

When atom 3 interacts resonantly with cavity 3, we switch the cavity off. Atom 3 is now sent to the Ramsey field which gives us the state

$$|\Psi_{a_1,a_2,a_3}(t)\rangle = \begin{array}{c} \frac{1}{2\sqrt{2}}|a\rangle\otimes\left[|P_{-2}^{(1)},P_{-2}^{(2)}\rangle + |P_0^{(1)},P_0^{(2)}\rangle - i|P_0^{(1)},P_{-2}^{(2)}\rangle + i|P_{-2}^{(1)},P_0^{(2)}\rangle\right] + \\ \frac{1}{2\sqrt{2}}|b\rangle\otimes\left[-i|P_0^{(1)},P_0^{(2)}\rangle + 1|P_0^{(1)},P_{-2}^{(2)}\rangle - 1|P_{-2}^{(1)},P_0^{(2)}\rangle - i|P_{-2}^{(1)},P_{-2}^{(2)}\rangle\right] \end{array}$$

Passing atom 3 through a Ramsey field,

$$|\Psi_{a_1,a_2,a_3}(t)\rangle =$$
$$\frac{1}{2\sqrt{2}}\frac{|b\rangle-|a\rangle}{\sqrt{2}}\otimes\left[|P_{-2}^{(1)},P_{-2}^{(2)}\rangle + |P_0^{(1)},P_0^{(2)}\rangle - i|P_0^{(1)},P_{-2}^{(2)}\rangle + i|P_{-2}^{(1)},P_0^{(2)}\rangle\right] +$$
$$\frac{1}{2\sqrt{2}}\frac{|b\rangle+|a\rangle}{\sqrt{2}}\otimes\left[-i|P_0^{(1)},P_0^{(2)}\rangle + 1|P_0^{(1)},P_{-2}^{(2)}\rangle - 1|P_{-2}^{(1)},P_0^{(2)}\rangle - i|P_{-2}^{(1)},P_{-2}^{(2)}\rangle\right]$$

After applying $P = |b\rangle\langle b|$, we get the following state which is independent of atom 3

$$|\Psi_{a_1,a_2}(t)\rangle = -\left(\frac{|P_0^{(1)}\rangle - i|P_{-2}^{(1)}\rangle}{\sqrt{2}}\right)\left(\frac{i|P_0^{(2)}\rangle - |P_{-2}^{(2)}\rangle}{\sqrt{2}}\right)$$

6.2.3 Case III

If, however, the cavity was initially in the Fock state $|1\rangle$, we can start with a momentum state of

$$|\Psi_{a_1}^{c_3}(t)\rangle = e^{-i\frac{P_{01}^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left[C_{P_l,a_1,c_2}^{a,n-1}(t)|a,0,P_l\rangle + C_{P_l,a_1,c_2}^{b,n}(t)|b,1,P_l\rangle\right]$$

This is the same equation as Eq. (3) for $n = 1$. Thus, solutions in the adiabatic approximation are

$$|\Psi_{a_1}^{c_3}(t)\rangle = \left[C_{P_0,a_1,c_3}^{b,1}(0)\cos\left(\frac{g_1^2}{4\Delta_1}t\right) + iC_{P_{-2},a_1,c_3}^{b,1}(0)\sin\left(\frac{g_1^2}{4\Delta_1}t\right)\right]|b,1,P_0\rangle +$$
$$\left[C_{P_{-2},a_1,c_3}^{b,1}(0)\cos\left(\frac{g_1^2}{4\Delta_1}t\right) + iC_{P_0,a_1,c_3}^{b,1}(0)\sin\left(\frac{g_1^2}{4\Delta_1}t\right)\right]|b,1,P_{-2}\rangle$$

From $|1\rangle \otimes \left(\frac{|P_0\rangle + i|P_{-2}\rangle}{\sqrt{2}}\right)$, we have $C_{P_0,a_1,c_3}^{b,1}(0) = -iC_{P_{-2},a_1,c_3}^{b,1}(0) = \frac{1}{\sqrt{2}}$ and $\frac{g_1^2}{4\Delta_1}t = \pi/2$

$$|\Psi_{a_1}(t)\rangle = \frac{-|P_0\rangle + i|P_{-2}\rangle}{\sqrt{2}}$$

The cavity is not entangled with the atom. Repeating the step for atom 2 and using the probabilities from $|1\rangle \otimes \left(\frac{i|P_0\rangle - |P_{-2}\rangle}{\sqrt{2}}\right)$ and $\frac{g_1^2}{4\Delta_1}t = \pi/2$, we get

$$|\Psi_{a_2}^{c_3}(t)\rangle = -\frac{i}{\sqrt{2}}|P_0\rangle - \frac{1}{\sqrt{2}}|P_{-2}\rangle$$

Again, the cavity is not entangled with atom 2. Atom 3 is now passed resonantly in cavity 3. We consider the same same initial state

$$|\Psi_{a_3}^{c_3}(t)\rangle = C_{a_3,c_3}^{a,0}(t)|a,0\rangle + C_{a_3,c_3}^{b,1}(t)|b,1\rangle$$

with a Hamiltonian

$$\mathcal{H} = \hbar g_3\left(\sigma_+\hat{a} + \sigma_-\hat{a}^\dagger\right)$$

From the probability amplitude method,

$$i\dot{C}_{a_3,c_3}^{a,0}(t)|a,0\rangle + i\dot{C}_{a_3,c_3}^{b,1}(t)|b,1\rangle$$
$$= g_3C_{a_3,c_3}^{b,1}(t)|a,0\rangle + g_3C_{a_3,c_3}^{a,0}(t)|b,1\rangle$$

$\Rightarrow$

$$i\dot{C}_{a_3,c_3}^{a,0}(t) = g_3C_{a_3,c_3}^{b,1}(t) \tag{32}$$

and

$$i\dot{C}_{a_3,c_3}^{b,1}(t) = g_3C_{a_3,c_3}^{a,0}(t) \tag{33}$$

Taking the Laplace transform of both the equations, we get

$$isF_{a_3,c_3}^{a,0}(s) - iC_{a_3,c_3}^{a,0}(0) = g_3F_{a_3,c_3}^{b,1}(s)$$

and

$$isF_{a_3,c_3}^{b,1}(s) - iC_{a_3,c_3}^{b,1}(0) = g_3F_{a_3,c_3}^{a,0}(s)$$

Arranging the Laplace Transform of the first equation, i.e. Eq (32), we get

$$F_{a_3,c_3}^{a,0}(s) = \frac{g_3F_{a_3,c_3}^{b,1}(s) + iC_{a_3,c_3}^{a,0}(0)}{is}$$

which can be put in the 2nd equation to get

$$isF^{b,1}_{a_3,c_3}(s) - iC^{b,1}_{a_3,c_3}(0) = g_3\frac{g_3F^{b,1}_{a_3,c_3}(s) + iC^{a,0}_{a_3,c_3}(0)}{is}$$

$$isF^{b,1}_{a_3,c_3}(s) = iC^{b,1}_{a_3,c_3}(0) + \frac{g_3^2F^{b,1}_{a_3,c_3}(s) + ig_3C^{a,0}_{a_3,c_3}(0)}{is}$$

$$isF^{b,1}_{a_3,c_3}(s) = iC^{b,1}_{a_3,c_3}(0) - i\frac{g_3^2F^{b,1}_{a_3,c_3}(s) + ig_3C^{a,0}_{a_3,c_3}(0)}{s}$$

$$is^2F^{b,1}_{a_3,c_3}(s) = isC^{b,1}_{a_3,c_3}(0) - ig_3^2F^{b,1}_{a_3,c_3}(s) - g_3C^{a,0}_{a_3,c_3}(0)$$

$$s^2F^{b,1}_{a_3,c_3}(s) = sC^{b,1}_{a_3,c_3}(0) - g_3^2F^{b,1}_{a_3,c_3}(s) + ig_3C^{a,0}_{a_3,c_3}(0)$$

$$\left[s + g_3^2\right]F^{b,1}_{a_3,c_3}(s) = sC^{b,1}_{a_3,c_3}(0) + ig_3C^{a,0}_{a_3,c_3}(0)$$

$$F^{b,1}_{a_3,c_3}(s) = \frac{s}{s^2+g_3^2}C^{b,1}_{a_3,c_3}(0) + i\frac{g_3}{s^2+g_3^2}C^{a,0}_{a_3,c_3}(0)$$

$\mathcal{L}\left[e^{-at}Sin(bt)\right] = \frac{b}{(s+a)^2+b^2}$ and $\mathcal{L}\left[e^{-at}Cos(bt)\right] = \frac{s+a}{(s+a)^2+b^2}$

$$C^{b,1}_{a_3,c_3}(t) = C^{b,1}_{a_3,c_3}(0)\cos(g_3t) + iC^{a,0}_{a_3,c_3}(0)\sin(g_3t)$$

From Eq. (33)

$$i\dot{C}^{b,1}_{a_3,c_3}(t) = g_3C^{a,0}_{a_3,c_3}(t)$$

$$C^{a,0}_{a_3,c_3}(t) = -C^{a,0}_{a_3,c_3}(0)\cos(g_3t) - iC^{b,1}_{a_3,c_3}(0)\sin(g_3t)$$

From $|b\rangle \otimes |1\rangle$, we have $C^{b,1}_{a_3,c_3}(0) = 1$ and $C^{a,0}_{a_3,c_3}(t) = 0$ and $g_3t = \pi/2$ which changes the state $|\Psi^{c_3}_{a_3}(t)\rangle$ to

$$|\Psi^{c_3}_{a_3}(t)\rangle = -i|a,0\rangle$$

The cavity switches off but the atom goes in the excited state. After passing it through a Ramsey field, $|\Psi^{c_3}_{a_3}(t)\rangle = -i\left(\frac{|b\rangle - |a\rangle}{\sqrt{2}}\right)|0\rangle$. If $P = |b\rangle\langle b|$ is applied after the interaction, atom 3 will go back to its state. This means that we end up with

$$|\Psi_{a_1,a_2}(t)\rangle = -\left(\frac{|P_0^{(1)}\rangle + i|P_{-2}^{(1)}\rangle}{\sqrt{2}}\right)\left(\frac{i|P_0^{(2)}\rangle - i|P_{-2}^{(2)}\rangle}{\sqrt{2}}\right)$$

6.2.4 Case IV

In the last case, we can consider the cavity initially in the state $|0\rangle$. The state $e^{-i\frac{P_{01}^2}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}C^{b,n}_{P_l,a_1,c_2}(t)|b,0,P_l\rangle$ on which Hamiltonian $\mathcal{H} = \frac{P_{x_1}^2}{2M_1} + \frac{\hbar\Delta_1}{2}\sigma_z + \hbar g_1Cos\left(kx_1\right)\left(\sigma_+\hat{a} + \sigma_-\hat{a}^\dagger\right)$ acts will produce the same equations for $n = 0$. Any $n = -1$ term will be automatically identically equal to 0 because there is no energy beyond the vacuum energy. Using the probability

amplitude method, we get $\dot{C}^{b,0}_{P_l,a_1,c_1}(t) = \left(\frac{l(l+l_0)\hbar k^2_{x_1}}{2M_1} - \frac{\Delta_1}{2}\right) C^{b,0}_{P_l,a_1,c_1}(t)$.and $C^{b,0}_{P_{l+1},a_1,c_2}(t) = C^{b,0}_{P_{l-1},a_1,c_2}(t) = 0$ (there is no change in momentum of the atom if there is no photon in the cavity). For $l = 0$ and $l = -2$, $C^{b,0}_{P_0,a_2,c_2}(t) = C^{b,0}_{P_0,a_2,c_2}(0) = 1$ and $C^{b,0}_{P_{-2},a_2,c_2}(t) = C^{b,0}_{P_{-2},a_2,c_2}(0) = 0$. Similarly $C^{b,0}_{P_{-2},a_3,c_2}(t) = 1$ whereas $C^{b,0}_{P_0,a_2,c_2}(t) = 0$ for atom 2. When atom 3 is sent to interact resonantly in this cavity, we have, for $|\Psi^{c_3}_{a_3}(t)\rangle = C^{b,0}_{a_3,c_3}(t)|b,0\rangle$ and $\mathcal{H} = \hbar g_3\left(\sigma_+\hat{a} + \sigma_-\hat{a}^\dagger\right)$, $\dot{C}^{b,0}_{a_3,c_3}(t) = 0 \Rightarrow C^{b,0}_{a_3,c_3}(t) = 1$. This reduces the vector $|\Psi^{c_3}_{a_3}(t)\rangle$ to $|b,0\rangle$. Applying $P = |b\rangle\langle b|$ will not change atom 3. Hence there is absolutely no change for in U_f if the cavity initially has a Fock state of $|0\rangle$. The atom will experience no change in momentum and $|\Psi_{a_1,a_2}(t)\rangle = \left(\frac{|P^{(1)}_0\rangle + i|P^{(1)}_{-2}\rangle}{\sqrt{2}}\right)\left(\frac{i|P^{(2)}_0\rangle - |P^{(2)}_{-2}\rangle}{\sqrt{2}}\right)$

6.3 Step 3

Now, atom 1 is passed through a final resonant cavity, cavity 4. The Hamiltonian works out in the same way on

$$|\Psi^{c_4}_{a_1,a_2}(t)\rangle = e^{-i\frac{P^2_{0_1}}{2M_1}t/\hbar}\sum_{l=-\infty}^{\infty}\left[C^{a,n-1}_{P_l,a_1,c_1}(t)|a,n-1,P_l\rangle + C^{b,n}_{P_l,a_1,c_1}(t)|b,n,P_l\rangle\right]$$

. The momentum states of atom 2 have not been stated. This is reasonable because the atoms are not entangled. The resulting rate equations are

$$\begin{aligned}
i\dot{C}^{a,n-1}_{P_l,a_1,c_4}(t) &= \left(\frac{l(l+l_0)\hbar k^2_{x_1}}{M_1} + \frac{\Delta_1}{2}\right) C^{a,n-1}_{P_l,a_1,c_1}(t) + \\
&\quad \frac{g_1\sqrt{n}}{2}\left(C^{b,n}_{P_{l+1},a_1,c_1}(t) + C^{b,n}_{P_{l-1},a_1,c_1}(t)\right) \\
i\dot{C}^{b,n}_{P_l,a_1,c_4}(t) &= \left(\frac{l(l+l_0)\hbar k^2_{x_1}}{2M_1} - \frac{\Delta_1}{2}\right) C^{b,n}_{P_l,a_1,c_1}(t) + \\
&\quad \frac{g_1\sqrt{n}}{2}\left(C^{a,n-1}_{P_{l-1},a_1,c_1}(t) + C^{a,n-1}_{P_{l+1},a_1,c_1}(t)\right)
\end{aligned}$$

Then, the solution of the state

$$|\Psi^{c_4}_{a_1,a_2}(t)\rangle = C^{b,n}_{P_{-2},a_1,c_4}(t)|b,n,P_{-2}\rangle + C^{b,n}_{P_0,a_1,c_4}(t)|b,n,P_0\rangle$$

with the phase factors ignored, is given by

$$\begin{aligned}
C^{b,n}_{P_{-2},a_1,c_4}(t) &= C^{b,n}_{P_{-2},a_1,c_4}(0)\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right) + iC^{b,n}_{P_0,a_1,c_4}(0)\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right) \\
C^{b,n}_{P_0,a_1,c4}(t) &= C^{b,n}_{P_0,a_1,c_4}(0)\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right) + iC^{b,n}_{P_{-2},a_1,c_4}(0)\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right)
\end{aligned}$$

The Hadamard operation is being performed only on the first atom. Hence, only probability amplitudes for atom 1 will be applied

6.3.1 Case I

If we continue with Case I, then from $\frac{|P_0^{(1)}\rangle - i|P_{-2}^{(1)}\rangle}{\sqrt{2}}$ for atom 1

$$-\frac{i}{\sqrt{2}}\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_{-2}\rangle + \frac{i}{\sqrt{2}}\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_{-2}\rangle + \frac{1}{\sqrt{2}}\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_0\rangle + \frac{1}{\sqrt{2}}\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_0\rangle$$

For $\frac{g_1^2 n}{4\Delta_1}t = 7\pi/4$, we get

$$|\Psi_{a_1}(t)\rangle = -i|P_{-2}\rangle$$

6.3.2 Case II

In this case, our initial conditions come from the state that has the form $|\Psi_{a_1,a_2}(t)\rangle = -\left(\frac{|P_0^{(1)}\rangle - i|P_{-2}^{(1)}\rangle}{\sqrt{2}}\right)\left(\frac{i|P_0^{(2)}\rangle + |P_{-2}^{(2)}\rangle}{\sqrt{2}}\right)$. This changes the state $|\Psi_{a_1,a_2}(t)\rangle$ to

$$\frac{i}{\sqrt{2}}\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_{-2}\rangle - \frac{i}{\sqrt{2}}\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_{-2}\rangle + \frac{-1}{\sqrt{2}}\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_0\rangle - \frac{1}{\sqrt{2}}\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_0\rangle$$

For $\frac{g_1^2 n}{4\Delta_1}t = 7\pi/4$

$$|\Psi_{a_1}(t)\rangle = i|P_{-2}\rangle$$

6.3.3 Case III

From $|\Psi_{a_1,a_2}(t)\rangle = -\left(\frac{|P_0^{(1)}\rangle + i|P_{-2}^{(1)}\rangle}{\sqrt{2}}\right)\left(\frac{i|P_0^{(2)}\rangle - |P_{-2}^{(2)}\rangle}{\sqrt{2}}\right)$ we place our initial conditions and get

$$-\frac{i}{\sqrt{2}}\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_{-2}\rangle + \frac{i}{\sqrt{2}}\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_{-2}\rangle + \frac{-1}{\sqrt{2}}\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_0\rangle - \frac{1}{\sqrt{2}}\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_0\rangle$$

For $\frac{g_1^2 n}{4\Delta_1}t = 7\pi/4$

$$|\Psi_{a_1}(t)\rangle = -|P_0\rangle$$

6.3.4 Case IV

From $|\Psi_{a_1,a_2}(t)\rangle = \left(\frac{|P_0^{(1)}\rangle + i|P_{-2}^{(1)}\rangle}{\sqrt{2}}\right)\left(\frac{i|P_0^{(2)}\rangle - |P_{-2}^{(2)}\rangle}{\sqrt{2}}\right)$, we have

$$\frac{i}{\sqrt{2}}\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_{-2}\rangle - \frac{i}{\sqrt{2}}\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_{-2}\rangle + \frac{1}{\sqrt{2}}\cos\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_0\rangle + \frac{1}{\sqrt{2}}\sin\left(\frac{g_1^2 n}{4\Delta_1}t\right)|b,n,P_0\rangle$$

For $\frac{g_1^2 n}{4\Delta_1}t = 7\pi/4$

$$|\Psi_{a_1}(t)\rangle = |P_0\rangle$$

6.4 Generalisation

Before the Hadamard is applied to the second atom in step 3, we have the following states, corresponding to each respective case:-

1. $\left(\frac{|P_0^{(1)}\rangle - i|P_{-2}^{(1)}\rangle}{\sqrt{2}}\right)\left(\frac{i|P_0^{(2)}\rangle - |P_{-2}^{(2)}\rangle}{\sqrt{2}}\right)$
2. $-\left(\frac{|P_0^{(1)}\rangle - i|P_{-2}^{(1)}\rangle}{\sqrt{2}}\right)\left(\frac{i|P_0^{(2)}\rangle - |P_{-2}^{(2)}\rangle}{\sqrt{2}}\right)$
3. $-\left(\frac{|P_0^{(1)}\rangle + i|P_{-2}^{(1)}\rangle}{\sqrt{2}}\right)\left(\frac{i|P_0^{(2)}\rangle - |P_{-2}^{(2)}\rangle}{\sqrt{2}}\right)$
4. $\left(\frac{|P_0^{(1)}\rangle + i|P_{-2}^{(1)}\rangle}{\sqrt{2}}\right)\left(\frac{i|P_0^{(2)}\rangle - |P_{-2}^{(2)}\rangle}{\sqrt{2}}\right)$

For the first atom, case I corresponds to $\left(\frac{(1-f(0)-f(1))|P_0^{(1)}\rangle + (f(1)-f(0))|P_{-2}^{(1)}\rangle}{\sqrt{2}}\right)$ for $f(0) = 0$ and $f(1) = 1$ i.e. for a balance f. The unit imaginary is a global resulting phase, which can be ignored for measurement purposes. If $f(0) = 1$ and $f(1) = 0$, then this corresponds to the second case. On the other hand, if $f(0) = f(1) = 1$, then we have our third case. If the function is balanced otherwise, then we have our fourth case. If m such atoms are passed in cavity 1 and cavity 3 has a generalised corresponding GHZ state initially, we get our Deutsch-Jozsa algorithm generalised.

7 Conclusion

The theory of quantum mechanics, developed as a limiting case to classical mechanics, notwithstanding its interpretive difficulties, has with it the elegance for paving way to a variety of applications. One such application is the implementation of a working Quantum Computer. The push one receives for using quantum principles as a measure of information and execution of algorithms is from quantum parallelism. It seems as though nature hides its enormous calculations. One such realisation of the power of Quantum Parallelism can be seen with quantum optics when one considers engineering a Quantum Computer, choosing techniques of cavity QED amongst many other competitors. The Deutsch-Josza algorithm, although of little practical significance, is an encouraging example which greatly reduces the time required for a specific function to be determined completely, when compared with its classical counterpart. The Hadamard gate has been physically realised, and so has the other unitary transformations in the Deutsch-Jozsa algorithm using different times of interactions in the cavity. Also, a generalisation of the Deutsch-Jozsa algorithm has been discussed, which might pave way for a working model of a Quantum Computer.

8 Appendix

8.1 Mathematical Spaces

Technically, a space is any set together with a well-defined structure on it.

Definition 2 *A vector space V over $\mathbb{F}$ endowed with the operation $\langle .,. \rangle : V \times V \longrightarrow \mathbb{F}$, called the inner product, is called an **inner product space** or pre-Hilbert space if $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha \in \mathbb{F}$ it satisfies the following axioms[101]:-*

- $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
- $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
- $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
- $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$

An inner product space V is compactly written as $(V, \langle .,. \rangle)$[101]. This notation may be confused with that for an ordered pair. However, one need not frustrate oneself over the matter since the use will be clear from context.

Example 3 *For $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle = \langle (x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \rangle = \sum_{i=1}^{n} \bar{x}_i y_i$ satisfies the above conditions and hence is an inner product space.[101]*

Proposition 4 *Let $(X, \langle .,. \rangle)$ be an inner product space. Then, for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in X$ over $\mathbb{F}$ and $\alpha, \beta \in \mathbb{F}$, the following properties hold[101]:-*

- $\langle \alpha \mathbf{v}_1 + \beta \mathbf{v}_2, \mathbf{v}_3 \rangle = \alpha \langle \mathbf{v}_1, \mathbf{v}_3 \rangle + \beta \langle \mathbf{v}_2, \mathbf{v}_3 \rangle$
- $\langle \mathbf{v}_1, \alpha \mathbf{v}_2 \rangle = \bar{\alpha} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$
- $\langle \mathbf{v}_1, \alpha \mathbf{v}_2 + \beta \mathbf{v}_3 \rangle = \bar{\alpha} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle + \bar{\beta} \langle \mathbf{v}_1, \mathbf{v}_3 \rangle$

Proof. For i)
$\langle \alpha \mathbf{v}_1 + \beta \mathbf{v}_2, \mathbf{v}_3 \rangle = \langle \alpha \mathbf{v}_1, \mathbf{v}_3 \rangle + \langle \beta \mathbf{v}_2, \mathbf{v}_3 \rangle = \alpha \langle \mathbf{v}_1, \mathbf{v}_3 \rangle + \beta \langle \mathbf{v}_2, \mathbf{v}_3 \rangle$
For ii), we have
$\langle \mathbf{v}_1, \alpha \mathbf{v}_2 \rangle = \overline{\langle \alpha \mathbf{v}_2, \mathbf{v}_1 \rangle} = \overline{\alpha \langle \mathbf{v}_2, \mathbf{v}_1 \rangle} = \bar{\alpha} \overline{\langle \mathbf{v}_2, \mathbf{v}_1 \rangle} = \bar{\alpha} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$
and finally for iii) we have
$\langle \mathbf{v}_1, \alpha \mathbf{v}_2 + \beta \mathbf{v}_3 \rangle = \overline{\langle \alpha \mathbf{v}_2 + \beta \mathbf{v}_3, \mathbf{v}_1 \rangle} = \overline{\alpha \langle \mathbf{v}_2, \mathbf{v}_1 \rangle} + \overline{\beta \langle \mathbf{v}_3, \mathbf{v}_1 \rangle} = \bar{\alpha} \overline{\langle \mathbf{v}_2, \mathbf{v}_1 \rangle} + \bar{\beta} \overline{\langle \mathbf{v}_3, \mathbf{v}_1 \rangle} = \bar{\alpha} \langle \mathbf{v}_1, \mathbf{v}_2 \rangle + \bar{\beta} \langle \mathbf{v}_1, \mathbf{v}_3 \rangle$ ■

Lemma 5 *Let I be an inner product with a corresponding norm. For $\mathbf{x}, \mathbf{y} \in I$, the space satisfies the Cauchy-Schwartz inequality $|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle$[101]*

Proof. For $\mathbf{y} = \mathbf{0}, \langle \mathbf{x}, \mathbf{0} \rangle = \langle \mathbf{x}, \mathbf{x} - \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{x} \rangle = 0$
Let $\alpha \in \mathbb{F}$. For $\mathbf{y} \neq 0, \langle \mathbf{x} - \alpha\mathbf{y}, \mathbf{x} - \alpha\mathbf{y} \rangle \geq 0$
$\Rightarrow \langle \mathbf{x}, \mathbf{x} - \alpha\mathbf{y} \rangle - \alpha \langle \mathbf{y}, \mathbf{x} - \alpha\mathbf{y} \rangle \geq 0$
$\Rightarrow \langle \mathbf{x}, \mathbf{x} \rangle - \bar{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle - \alpha \left[\langle \mathbf{y}, \mathbf{x} \rangle - \bar{\alpha} \langle \mathbf{y}, \mathbf{y} \rangle \right] \geq 0$
For $\bar{\alpha} = \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}$, we have
$\langle \mathbf{x}, \mathbf{x} \rangle - \bar{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle - \alpha \left[\langle \mathbf{y}, \mathbf{x} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle \right] = \langle \mathbf{x}, \mathbf{x} \rangle - \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle} \langle \mathbf{x}, \mathbf{y} \rangle \geq 0$
$\Rightarrow \langle \mathbf{x}, \mathbf{x} \rangle - \frac{\overline{\langle \mathbf{x}, \mathbf{y} \rangle}}{\langle \mathbf{y}, \mathbf{y} \rangle} \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - \frac{|\langle \mathbf{x}, \mathbf{y} \rangle|^2}{\langle \mathbf{y}, \mathbf{y} \rangle} \geq 0$
$\Rightarrow \langle \mathbf{x}, \mathbf{x} \rangle \geq \frac{|\langle \mathbf{x}, \mathbf{y} \rangle|^2}{\langle \mathbf{y}, \mathbf{y} \rangle}$
$\Rightarrow \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle \geq |\langle \mathbf{x}, \mathbf{y} \rangle|^2$
$\Rightarrow |\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle$
If $\mathbf{x} = \alpha\mathbf{y}$, then
$\langle \mathbf{x} - \alpha\mathbf{y}, \mathbf{x} - \alpha\mathbf{y} \rangle = \langle \alpha\mathbf{y} - \alpha\mathbf{y}, \alpha\mathbf{y} - \alpha\mathbf{y} \rangle = \langle \mathbf{0}, \mathbf{0} \rangle = 0$
or $\langle \mathbf{x} - \alpha y, \mathbf{x} - \alpha\mathbf{y} \rangle = 0$
$\Rightarrow \langle \mathbf{x}, \mathbf{x} - \alpha\mathbf{y} \rangle - \alpha \langle \mathbf{y}, \mathbf{x} - \alpha\mathbf{y} \rangle = 0$
$\Rightarrow \langle \mathbf{x}, \mathbf{x} \rangle - \bar{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle - \alpha \left[\langle \mathbf{y}, \mathbf{x} \rangle - \bar{\alpha} \langle \mathbf{y}, \mathbf{x} \rangle \right] = 0$
Again, for $\bar{\alpha} = \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}$, we have
$\langle \mathbf{x}, \mathbf{x} \rangle - \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle} \langle \mathbf{x}, \mathbf{y} \rangle = 0$
$\Rightarrow \langle \mathbf{x}, \mathbf{x} \rangle - \frac{\overline{\langle \mathbf{x}, \mathbf{y} \rangle}}{\langle \mathbf{y}, \mathbf{y} \rangle} \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - \frac{|\langle \mathbf{x}, \mathbf{y} \rangle|^2}{\langle \mathbf{y}, \mathbf{y} \rangle} = 0$
$\Rightarrow \langle \mathbf{x}, \mathbf{x} \rangle = \frac{|\langle \mathbf{x}, \mathbf{y} \rangle|^2}{\langle \mathbf{y}, \mathbf{y} \rangle}$
$\Rightarrow |\langle \mathbf{x}, \mathbf{y} \rangle|^2 = \langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle$
i.e. equality will hold if the vectors are multiples of each other. ■

Definition 6 *Two elements* $\mathbf{x}, \mathbf{y}$ *of an inner product space are* ***orthogonal*** *if* $\langle \mathbf{x}, \mathbf{y} \rangle = 0$*.[101]*

This is written as $\mathbf{x} \perp \mathbf{y}$. If, furthermore, the norm of the two elements is 1, then the two are said to be orthonormal to each other. Two inner product spaces A and B are orthogonal if $\forall \mathbf{x} \in A$ and $\forall \mathbf{y} \in B$, $\mathbf{x} \perp \mathbf{y}$. This is written as $A \perp B$.

Theorem 7 *An orthonormal set is linearly independent[101]*

Proof. Let $\{e_1, e_2, ..., e_n\}$ be orthonormal. Consider $\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = 0$
Then, for any $j \in \{1, 2, ..., n\}$, $\left\langle \sum_{i=1}^{n} \alpha_i e_i, e_j \right\rangle = 0$
$\Rightarrow \sum_{i=1}^{n} \alpha_i \langle e_i, e_j \rangle = 0$
$\Rightarrow \alpha_j \langle e_j, e_j \rangle = 0$
$\Rightarrow \alpha_j = 0$
Since j is arbitrary, therefore $\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = 0$
$\Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_n = 0$ ■

Definition 8 *Any space* X *with a norm defined on it is called* ***complete*** *if every Cauchy sequence in* X *converges in* X*[101]*

Definition 9 *A complete inner product space is known as a* ***Hilbert Space****.[101]*

Example 10 *For* $1 \leq p \leq \infty, L^P(\Omega) = \{f : f \text{ is measurable on } \Omega \text{ and } \|f\|_P < \infty\}$ *where for* $1 \leq p < \infty$, $\|f\|_P = \left(\int_\Omega \|f\|^P \, dx\right)^{1/P}$ *and* $\|f\|_\infty = \underset{x}{Sup} \frac{|f(x)|}{\|x\|}$

L^p spaces, sometimes called Lebesgue spaces, are defined using natural generalisations of p-norms for finite-dimensional vector spaces. They are named after the French mathematician Henri Lebesgue (June 28, 1875 – July 26, 1941). It is from these spaces that Quantum Mechanics adopts its mechinary. $L^2(-\infty, \infty)$ is a particular example of a space of square-integrable function. It is important from an application point of view. Hilbert spaces are named after the German mathematician David Hilbert

Definition 11 *Let* H *be a Hilbert space and* $B \subset H$*.* B *is an* ***orthonormal system*** *of* H *if* $\|\mathbf{x}\| = 1 \ \forall \mathbf{x} \in B$ *and* $\mathbf{x} \perp \mathbf{y} \ \forall \mathbf{x}, \mathbf{y} \in B$*[101].*

This definition is employed when the orthonormal basis (axis) of any space need to be clarified. In this case and in the next few, notice that even the basis have been generalised

Definition 12 *Let* H_1 *and* H_2 *be two Hilbert spaces of dimension* n *and* k *respectively. Given two vectors* $(x_1, x_2, ..., x_n) \in H_1$ *and* $(y_1, y_2, ..., y_3) \in H_2$*. The tensor product* $\otimes$ *of* $\mathbf{x}$ *and* $\mathbf{y}$*, written compactly as* $\mathbf{x} \otimes \mathbf{y}$*, or even* $\mathbf{xy}$ *is defined as* $\mathbf{x} \otimes \mathbf{y} := (x_1y_1, x_1y_2, ..., x_1y_k, x_2y_1, x_2y_2, ..., x_2y_k, ..., x_ny_1, x_ny_2, ..., x_ny_k)$*[101].*

One can even take the tensor product of two spaces altogether to form a "bigger" space by taking the tensor of each element of the former space with each element of the latter space i.e. $H_1 \otimes H_2 = \{x \otimes y, x \in H_1, y \in H_2\}$. This construction accounts for a system of particles..If, however, one wishes to break a Hilbert space into its constituent orthogonal spaces, then one considers the direct sum of two spaces. If finitely many Hilbert spaces $H_1, H_2, ..., H_n$ are given, one can construct their direct sum. Formally, this is done as follows. Let H_1 and H_2 be two Hilbert spaces over a Field $\mathbb{F}$. The Cartesian product $H_1 \times H_2$ can be given a space structure by using the direct sum $H_1 \oplus H_2$ and then turn this into a Hilbert space by defining the inner product as

$$\langle (x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle + ... + \langle x_n, y_n \rangle$$

and

$$\alpha(\mathbf{x}, \mathbf{y}) = (\alpha\mathbf{x}, \alpha\mathbf{y})$$

Proposition 13 *Let* H_1, H_2, H_3 *be Hilbert spaces of dimension* n*,* k *and* m *respectively, over a field* $\mathbb{F}$*. For* $\alpha \in \mathbb{F}$*,* $\mathbf{y}, \mathbf{y}' \in H_1$*,* $\mathbf{x}, \mathbf{x}' \in H_2$ *and* $\mathbf{w} \in H_3$

- $(\mathbf{x} \otimes \mathbf{y}) \otimes \mathbf{w} = \mathbf{x} \otimes (\mathbf{y} \otimes \mathbf{w})$
- $\alpha(\mathbf{x} \otimes \mathbf{y}) = (\alpha\mathbf{x}) \otimes \mathbf{y} = x \otimes (\alpha\mathbf{y})$

- $(\mathbf{x}+\mathbf{x}')\otimes y = (\mathbf{x}\otimes\mathbf{y})+(\mathbf{x}'\otimes\mathbf{y})$
- $\mathbf{x}\otimes(\mathbf{y}+\mathbf{y}') = (\mathbf{x}\otimes\mathbf{y})+(\mathbf{x}\otimes\mathbf{y}')$

Proof. $(\mathbf{x}\otimes\mathbf{y})\otimes\mathbf{w}$
$= (x_1y_1, x_1y_2, ..., x_1y_k, x_2y_1, x_2y_2, ..., x_2y_k, ..., x_ny_1, x_ny_2, ..., x_ny_k)\otimes(w_1, w_2, ..., w_m)$
$= \mathbf{x}\otimes(\mathbf{y}\otimes\mathbf{w})$
Next, $\alpha(\mathbf{x}\otimes\mathbf{y})$
$= \alpha(x_1y_1, x_1y_2, ..., x_1y_k, x_2y_1, x_2y_2, ..., x_2y_k, ..., x_ny_1, x_ny_2, ..., x_ny_k)$
$= (\alpha x_1y_1, \alpha x_1y_2, ..., \alpha x_1y_k, \alpha x_2y_1, \alpha x_2y_2, ..., \alpha x_2y_k, ..., \alpha x_ny_1, \alpha x_ny_2, ..., \alpha x_ny_k)$
Then, $((\alpha x_1)\,y_1, ..., (\alpha x_1)\,y_k, (\alpha x_2)\,y_1, ..., (\alpha x_2)\,y_k, ..., \left(\alpha x_{(n)}\right)y_1, ..., \left(\alpha x_{(n)}\right)y_k) \Rightarrow (\alpha\mathbf{x})\otimes\mathbf{y}$
and $(x_1\,(\alpha y_1), ..., x_1\,(\alpha y_k), x_2\,(\alpha y_1), ..., x_2\,(\alpha y_k), ..., x_n\,(\alpha y_1), ..., x_n\,(\alpha y_k)) \Rightarrow \mathbf{x}\otimes(\alpha\mathbf{y})$
For the third proposition, $(\mathbf{x}+\mathbf{x}')\otimes\mathbf{y}$
$= (x_1y_1 + x_1'y_1, ..., x_1y_k + x_1'y_k, ..., x_ny_k + x_n'y_k)$
$= (x_1y_1, ..., x_1y_k, ..., x_ny_k) + (x_1'y_1, ..., x_1'y_k, ..., x_n'y_k)$
$= (\mathbf{x}\otimes\mathbf{y}) + (\mathbf{x}'\otimes\mathbf{y})$
The fourth proposition follows in a similar manner. ■

$H_1\otimes H_2$ is a Hilbert space. Firstly, the set of the tensor product of elements from two different fields yields a field itself. Secondly, one can construct a vector space from two others since every vector space is free and finally, one can define the inner product $\langle .,.\rangle_{H_1\otimes H_2} : H_1\otimes H_2 \longrightarrow \mathbb{F}_1\otimes\mathbb{F}_2$ by $\langle \mathbf{v}_1\otimes\mathbf{v}_2, \mathbf{u}_1\otimes\mathbf{u}_2\rangle_{H_1\otimes H_2} = \langle \mathbf{v}_1, \mathbf{u}_1\rangle\langle v_2, \mathbf{u}_2\rangle$ for $\mathbf{v}_1, \mathbf{u}_1 \in H_1$ and $\mathbf{v}_2, \mathbf{u}_2 \in H_2$. Completeness can be shown by using the same inner product. If H_1 and H_2 have orthonormal bases $\{e_n\}$ and $\{e\prime_k\}$, respectively, then $\{e_n\otimes e\prime_k\}$ is an orthonormal basis for $H_1\otimes H_2$. Furthermore, the dimension of $H_1\otimes H_2$ is the product (as cardinal numbers) of the Hilbert dimensions i.e. the dimension of such a space is $n\times k$.

Proof. Let $e_1, e_2, ..., e_n$ and $e_1', e_2', ..., e_k'$ be linearly independent basis for H_1 and H_2 respectively. Then, From the basis $(e_1, e_2, ..., e_n)\otimes(e_1', e_2', ..., e_k')$ we can form the sum $c_1c_1'e_1e_1' + c_1c_2'e_1e + ... + c_1c_k'e_1e_k' + c_2c_1'e_2e_1' + c_2c_2'e_2e_2' + ... + c_2c_k'e_2'e_k' + ... + c_nc_1'e_ne_1' + c_nc_2'e_ne_2' + ... + c_nc_k'e_n'e_k'$
If $(e_1, e_2, ..., e_n)\otimes(e_1', e_2', ..., e_k') = \mathbf{0}$
then $c_1e_1(c_1'e_1' + c_2'e_2' + ... + c_2'e_k') + c_2e_2(c_1'e_1' + c_2'e_2' + ... + c_2'e_k') + ... + c_ne_n(c_1'e_1' + c_2'e_2' + ... + c_2'e_k') = \mathbf{0}$
$\Rightarrow (c_1e_1 + c_2e_2 + ... + c_ne_n)\,(c_1'e_1' + c_2'e_2' + ... + c_2'e_k') = \mathbf{0}$
$\Rightarrow (c_1e_1 + c_2e_2 + ... + c_ne_n) = 0$ or $(c_1'e_1' + c_2'e_2' + ... + c_2'e_k') = \mathbf{0}$
$\Rightarrow c_1, c_2, ..., c_n = 0$ or $c_1', c_2', ..., c_n' = \mathbf{0}$
$\Rightarrow c_1c_1', c_1c_2', ..., c_1c_k', ..., c_nc_k' = \mathbf{0}$
$\Rightarrow (e_1, e_2, ..., e_n)\otimes(e_1', e_2', ..., e_k')$ is linearly independent
This completes the proof. ■

A dual space of any Hilbert space H is a collection of all functionals on H. The Hermitian of any vector is seen by applying the Hermitian operator on the vector with the property that $\langle T(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, T^*(\mathbf{y})\rangle = \langle T^*(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, T(\mathbf{y})\rangle$. The Hermitian conjugate of any vector in a Hilbert space H is an element of the dual space of H

8.2 Operators on a Hilbert Space

Definition 14 *An **operator** is a mapping from one space to another.*

For any operator A, B defined from a vector space U to vector space V over $\mathbb{F}$, operator addition and multiplication of operators is defined in the following way[101]

- $\forall \mathbf{x} \in U$, $(A+B)(\mathbf{x}) = A(\mathbf{x}) + B(\mathbf{x}) \in V$
- $\forall \alpha \in \mathbb{F}$, $(\alpha A)\mathbf{x} = \alpha\,(A\mathbf{x}) \in V$

The addition of operators is commutative. However, the products of operators is not always commutative. The product is, however, associative. A functional, on the other hand, is a mapping from a vector space to a field. Many properties of functionals are analogous to those of operators[101].

The linear operator merits special attention:–

Definition 15 *Let H_1, H_2 be two Hilbert spaces over $\mathbb{F}$. $T : H_1 \longrightarrow H_2$ is **linear** if, $\forall \mathbf{x}, \mathbf{y} \in H_1$ and $\forall \alpha \in \mathbb{F}$, $T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$ and $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$[101]*

Example 16 *An operator T from $C[a,b]$ into itself defined by*

$$T(x(t)) = tx(t)$$

Of special importance is the fact that for any operator T and $\mathbf{0}$ vector, $T(\mathbf{0}) = \mathbf{0}$

Proof. $T(\mathbf{0}) = T(\mathbf{x}-\mathbf{x})$
$= T(\mathbf{x} + (-1\mathbf{x})) = T(\mathbf{x}) + \mathbf{T}(-1\mathbf{x}) = T(\mathbf{x}) - \mathbf{T}(\mathbf{x})$
$= \mathbf{0}$ ■

Example 17 *Another operator T from $C[a,b]$ into itself can be defined by $T(x(t)) = \int_a^t x(\tau)d\tau$ is linear.*

Definition 18 *Let H_1, H_2 be Hilbert spaces and $T : H_1 \longrightarrow H_2$ be a linear operator. Then, the Hilbert **adjoint** operator T^*: $H_2 \longrightarrow H_1$ of T is such that, $\forall \mathbf{x} \in H_1$ and $\mathbf{y} \in H_2$*

$$\langle T(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, T^*(\mathbf{y})\rangle$$

Theorem 19 *The Hilbert adjoint operator T^* of T is unique*

Proof. Let ${T^*}_1 : H_2 \longrightarrow H_1$ and ${T^*}_2 : H_2 \longrightarrow H_1$ be Hilbert adjoints of $T : H_1 \longrightarrow H_2$

Then, $\forall \mathbf{x} \in H_1$ and $\mathbf{y} \in H_2$
$\langle T(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, {T^*}_1(\mathbf{y})\rangle = \langle \mathbf{x}, {T^*}_2(\mathbf{y})\rangle$
$\Rightarrow {T^*}_1(\mathbf{y}) = {T^*}_2(\mathbf{y})\ \forall \mathbf{y} \in H_2$
$\Rightarrow {T^*}_1 = {T^*}_2$ ■

Definition 20 *A bounded linear operator $T : H \longrightarrow H$ on a Hilbert space H is said to be*

- *self-adjoint or Hermitian if $T^* = T$*
- *unitary if T is bijective and $T^* = T^{-1}$*

Thus, if T is self-adjoint, then $\langle T(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, T(\mathbf{y})\rangle$[101]

Example 21 *The usual dot product $\langle ., .\rangle : \mathbb{C}^n \longrightarrow \mathbb{C}$ is defined as $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x^T y}$ where* $\mathbf{x}$,$\mathbf{y}$ *are column vectors.*

Definition 22 *Let T be an operator and* $\mathbf{x}$ *be an element of a Hilbert space. If there is a scalar α such that $T(\mathbf{x}) = \alpha\mathbf{x}$, then α is called the* ***eigenvalue*** *T corresponding to that* ***eigenvector***

Theorem 23 *The eigenvalues of every Hermitian operator are real*

Proof. Let H be a Hilbert space over $\mathbb{F}$ and $T(\mathbf{x}) = \alpha\mathbf{x}$ for $\alpha \in \mathbb{F}$ and $\mathbf{x} \in H$
Then, $\langle T(\mathbf{x}), \mathbf{y}\rangle = \langle T^*(\mathbf{x}), \mathbf{y}\rangle$
$\Rightarrow \langle \alpha\mathbf{x}, \mathbf{y}\rangle = \langle \bar{\alpha}\mathbf{x}, \mathbf{y}\rangle$
$\Rightarrow \alpha \langle \mathbf{x}, \mathbf{y}\rangle = \bar{\alpha} \langle \mathbf{x}, \mathbf{y}\rangle$
$\Rightarrow \alpha = \bar{\alpha}$ if $\langle \mathbf{x}, \mathbf{y}\rangle \neq 0$ ■

Definition 24 *Let H be a Hilbert space and $U \subset H$. An operator $\hat{P} : H \longrightarrow U$ is called the* ***projection operator*** *if $\hat{P}^\dagger = \hat{P}$ and $\hat{P}^2 = \hat{P}$.*

The product of two commuting projection operators is also a projection operator.

Proof. Let $\hat{P}_1$ and $\hat{P}_2$ be two projection operators. Then,
$\left(\hat{P}_1\hat{P}_2\right)^\dagger = \hat{P}_2^\dagger\hat{P}_1^\dagger$
$= \hat{P}_2\hat{P}_1 = \hat{P}_1\hat{P}_2$
And $\left(\hat{P}_1\hat{P}_2\right)^2 = \hat{P}_1\hat{P}_2\hat{P}_1\hat{P}_2$
$= \hat{P}_1\hat{P}_1\hat{P}_2\hat{P}_2 = \hat{P}_1^2\hat{P}_2^2 = \hat{P}_1\hat{P}_2$ ■

The sum of two projection operators is not necessarily a projection operator itself. Two projection operators are orthogonal if their product is zero. Thus, $\hat{P}_j\hat{P}_i = \delta_{ij}\hat{P}_i$ where

$$\delta_{ij} = \begin{array}{ll} 1 & i = j \\ 0 & i \neq j \end{array}$$

is the Kronecker-Delta "function". The sum of two projection operators is a projection operator if and only if the projection operators are mutually orthogonal

Proof. $\left(\hat{P}_1 + \hat{P}_2\right)^\dagger = \hat{P}_1^\dagger + \hat{P}_2^\dagger = \hat{P}_1 + \hat{P}_2$
And $\left(\hat{P}_1 + \hat{P}_2\right)^2 = \left(\hat{P}_1 + \hat{P}_2\right)\left(\hat{P}_1 + \hat{P}_2\right)$
$= \hat{P}_1^2 + \hat{P}_1\hat{P}_2 + \hat{P}_2\hat{P}_1 + \hat{P}_2^2$
$= \hat{P}_1 + \hat{P}_2$ ■

8.3 Laplace Transform

Definition 25 *Let $f(t)$ be a function. The* ***Laplace transform*** *of f, denoted by $\mathcal{L}[f(t)]$, is defined as* $\int_0^\infty e^{-st} f(t)dt$

This transformation is linear

Proof. $\mathcal{L}[\alpha f(t)+\beta g(t)] = \int_0^\infty e^{-st}\left(\alpha f(t) + \beta g(t)\right) dt = \int_0^\infty e^{-st}\alpha f(t)dt + \int_0^\infty e^{-st}\beta g(t)dt$

$= \alpha\int_0^\infty e^{-st} f(t)dt + \beta\int_0^\infty e^{-st} g(t)dt = \alpha\mathcal{L}[f(t)] + \beta\mathcal{L}[g(t)]$ ■

By convention the Laplace transformation of a derivative is denoted by $\mathcal{L}[f(t)] = F(s)$. The inverse Laplace transform of a known identity can be deduced from a table of values. Evaluating a Laplace inverse without any table is beyond the scope of this thesis.

Here are a few useful properties stated as lemmas directly from the above definition

Lemma 26 $\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$

Proof. $\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$= \frac{f(t)e^{-st}}{-s}\Big]_0^\infty - \int^\infty \frac{e^{-st}}{-s} f'(t)dt$

$= \frac{f(0)}{-s} + \frac{1}{s}\mathcal{L}[f'(t)]$ for $s > 0$

i.e. $\mathcal{L}[f(t)] = \frac{f(0)}{-s} - \frac{1}{s}\mathcal{L}[f'(t)]$ for $s > 0$

or $s\mathcal{L}[f(t)] - f(0) = \mathcal{L}[f'(t)]$ for $s > 0$ ■

Lemma 27 $\mathcal{L}[e^{-at} f(t)] = F(s + a)$

Proof. $\mathcal{L}[e^{-at} f(t)] = \int_0^\infty e^{-st} e^{-at} f(t)dt$

$= \int_0^\infty e^{-(s+a)t} f(t)dt$

Let $s + a = v$. Then, $\int_0^\infty e^{-(s+a)t} f(t)dt = \int_0^\infty e^{-vt} f(t)dt$

$= F(v) = F(s + a)$ ■

Lemma 28 $\mathcal{L}\left[Sin(bt)\right] = \frac{b}{s^2+b^2}$

Proof. $\mathcal{L}\left[Sin(bt)\right] = \int_0^\infty e^{-st} Sin(bt)dt$

$= \int_0^\infty e^{-st}\frac{\left[e^{ibt} - e^{-ibt}\right]}{2i}dt = \frac{1}{2i}\int_0^\infty e^{-(s-ib)t}dt - \frac{1}{2i}\int_0^\infty e^{-(s+ib)t}dt$

$= \frac{1}{2i}\frac{e^{-(s-ib)t}}{-(s-ib)}\Big]_0^\infty - \frac{1}{2i}\frac{e^{-(s+ib)t}}{-(s+ib)}\Big]_0^\infty$

$= \frac{1}{2i}\left[\frac{1}{(s-ib)} - \frac{1}{(s+ib)}\right]$ for $s > 0$
$= \frac{b}{(s-ib)(s+ib)} = \frac{b}{s^2+b^2}$ ■

From this lemma and the one that follows, one may easily determine Laplace inverse of algebraic expressions of the forms written in the lemmae.

Lemma 29 $\mathcal{L}[Cos(bt)] = \frac{s}{s^2+b^2}$

Proof. $\mathcal{L}[Cos(bt)] = \int_0^\infty e^{-st} Cos(bt)dt$
$= \int_0^\infty e^{-st}\frac{\left[e^{ibt}+e^{-ibt}\right]}{2}dt = \frac{1}{2}\int_0^\infty e^{-(s-ib)t}dt + \frac{1}{2}\int_0^\infty e^{-(s+ib)t}dt$
$= \frac{1}{2}\frac{e^{-(s-ib)t}}{-(s-ib)}\Big]_0^\infty + \frac{1}{2}\frac{e^{-(s+ib)t}}{-(s+ib)}\Big]_0^\infty$
$= \frac{1}{2}\left[\frac{1}{(s-ib)} + \frac{1}{(s+ib)}\right]$ for $s > 0$
$= \frac{1}{2}\frac{2s}{(s-ib)(s+ib)} = \frac{s}{s^2+b^2}$ ■

Corollary 30 $\mathcal{L}[e^{-at}\,\mathrm{Si}\,n(bt)] = \frac{b}{(s+a)^2+b^2}$ *and* $\mathcal{L}[e^{at}Cos(bt)] = \frac{s+a}{(s+a)^2+b^2}$

8.4 Dirac Formalism

The postulates mentioned in the thesis presuppose a continuous nature of the problem at hand. This is the Schrodinger formulation. Hiesenberg had his own formulation of quantum mechanics and this involved various matrices. This has been developed in to a fascinating Density Operator approach, which can treat an ensemble of states in a neat manner. Later in time, formulations involving matrices and waves were shown to be equal[82]. To reconcile with the two opposing ideas, Paul Andre Maurice Dirac introduced his formulation of quantum mechanics in 1930[102], involving bras and kets. Of the various ways to represent a quantum system, the bra-ket notation has been chosen as the standard notation because of its flexibility. A bra is represented by the symbol $\langle .|$ and a ket by $|.\rangle$ which, when combined, represent the inner product $\langle .|.\rangle$, to form an element of the field $\mathbb{F}$ the Hilbert space scales over. Here, the separation is represented by a vertical line instead of a comma. Now, a state is a vector $|\psi\rangle$ in a Hilbert space. The vector may even have parameters to determine its state. Instead of the usual functional notation, the operator notation takes the form $\hat{H}|\psi\rangle$. If $\hat{H}|\psi\rangle = |\phi\rangle$, then $\langle\psi|\hat{H}|\psi\rangle = \langle\psi|\phi\rangle$. For two states $|\psi\rangle$ and $|\phi\rangle$, the inner product $\langle\phi|\psi\rangle$ represents the state of the vector $|\psi\rangle$ after the operation of $\langle\phi|$. The concept of an eigenvector and eigenvalue carry over. The normalisation condition is replaced by $|\langle\psi|\psi\rangle|^2 = 1$. Notice that the calculation of the integral of a complex-valued function is replaced by an inner product. A ket is a column vector in a Hilbert Space of its Hermitian conjugate, bra, and vice versa. i.e. $(|\alpha\psi\rangle)^\dagger = (\alpha|\psi\rangle)^\dagger = \bar{\alpha}\langle\psi|$. Each bra has only one corresponding ket and vice versa.

8.5 Probability Theory

Definition 31 *Let x be a random variable.* ***Probability density function*** *of x is a function that describes the probability of that random variable to occur at a given point.*

The probability for a random variable to occur within a given region is given by the integral of the function between the two points. The probability density function may take on values greater than one.

Definition 32 *Let x be a random variable which can take on a probability p_1 at x_2, p_2 at x_2 and so on. Then, the* ***expectation value*** *of x, written $\langle x \rangle$ is defined as $\sum_i x_i p_i$*

Analogously, if a random variable x admits a distribution defined by a probability density function $f(x)$, then the expectation value can be calculated by $\int_{-\infty}^{\infty} x f(x) dx$. In quantum mechanics, this is not the most probable value. Like classical statistics, it only tells one of the mean value that will be obtained when number of measurements becomes very large. This can be further generalised to include the expectation values of an operator $\hat{X}$ on a wave function $\psi(x)$; $\left\langle \hat{X} \right\rangle := \int_{-\infty}^{\infty} \hat{X} \left|\psi\left(x\right)\right|^2 dx$. In the Dirac Notation, this is equivalent to $\left\langle \psi \left| \hat{X} \right| \psi \right\rangle$

8.6 Wave equation in a homogenous medium

Definition 33 *The **nabla** symbol* ∇ *on* $\mathbb{R}^3$ *is defined as* $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$

Proposition 34 *Let* $\mathbf{V} = (v_1, v_2, v_3) \in \mathbb{R}^3$*, where each coordinate is a function of* x*,* y*,* z*. Then,* $\nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla.\mathbf{V}) - \nabla^2 \mathbf{V}$

Proof. $\nabla \times (\nabla \times \mathbf{V}) = \nabla \times \left[(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (v_1, v_2, v_3) \right]$

$= \nabla \times \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)$

$= \left(\frac{\partial^2 v_2}{\partial yx} + \frac{\partial^2 v_3}{\partial xz} - \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial^2 v_1}{\partial z^2}, \frac{\partial^2 v_2}{\partial yz} + \frac{\partial^2 v_1}{\partial xy} - \frac{\partial^2 v_2}{\partial z^2} - \frac{\partial^2 v_2}{\partial x^2}, \frac{\partial^2 v_1}{\partial xz} + \frac{\partial^2 v_2}{\partial zy} - \frac{\partial^2 v_3}{\partial z^2} - \frac{\partial^2 v_3}{\partial y^2} \right)$

$$= \begin{pmatrix} \frac{\partial^2 v_2}{\partial yx} + \frac{\partial^2 v_3}{\partial xz} + \frac{\partial^2 v_1}{\partial x^2} - \frac{\partial^2 v_1}{\partial x^2} - \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial^2 v_1}{\partial z^2} \\ \frac{\partial^2 v_2}{\partial yz} + \frac{\partial^2 v_1}{\partial xy} + \frac{\partial^2 v_2}{\partial y^2} - \frac{\partial^2 v_2}{\partial x^2} - \frac{\partial^2 v_2}{\partial y^2} - \frac{\partial^2 v_2}{\partial z^2} \\ \frac{\partial^2 v_1}{\partial xz} + \frac{\partial^2 v_2}{\partial zy} + \frac{\partial^2 v_3}{\partial x^2} - \frac{\partial^2 v_3}{\partial x^2} - \frac{\partial^2 v_3}{\partial y^2} - \frac{\partial^2 v_3}{\partial z^2} \end{pmatrix}^T$$

$$= \begin{pmatrix} \frac{\partial^2 v_2}{\partial yx} + \frac{\partial^2 v_3}{\partial xz} + \frac{\partial^2 v_1}{\partial x^2} \\ \frac{\partial^2 v_3}{\partial yz} + \frac{\partial^2 v_1}{\partial xy} + \frac{\partial^2 v_2}{\partial y^2} \\ \frac{\partial^2 v_1}{\partial xz} + \frac{\partial^2 v_2}{\partial zy} + \frac{\partial^2 v_3}{\partial z^2} \end{pmatrix}^T - \begin{pmatrix} \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \\ \frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} + \frac{\partial^2 v_2}{\partial z^2} \\ \frac{\partial^2 v_3}{\partial x^2} + \frac{\partial^2 v_3}{\partial y^2} + \frac{\partial^2 v_3}{\partial z^2} \end{pmatrix}^T$$

The term on the left becomes

$$\left(\frac{\partial}{\partial x} \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right), \frac{\partial}{\partial y} \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right), \frac{\partial}{\partial z} \left(\frac{\partial v_1}{\partial z} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) \right)$$

$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right)$

$= \nabla (\nabla.(v_1, v_2, v_3))$

$= \nabla (\nabla.\mathbf{V})$

The one on the right becomes

$$\left(\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) v_1, \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) v_2, \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) v_3 \right)$$

$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (v_1, v_2, v_3)$

$= \left[\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) . \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \right] (v_1, v_2, v_3)$

$= \nabla^2 \mathbf{V}$

This completes the proof ■

Among the four famous Maxwell's equations listed in section V, we start from $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ for a source free region. Applying the nabla operator on

both sides, we get

$$\nabla \times (\nabla \times \mathbf{E}) = -\boldsymbol{\nabla} \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$
$$\nabla (\nabla . \mathbf{E}) - \nabla^2 \mathbf{E} = -\boldsymbol{\mu}_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{H})$$
$$0 - \nabla^2 \mathbf{E} = -\boldsymbol{\mu}_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{H})$$
$$\nabla^2 \mathbf{E} = \boldsymbol{\mu}_0 \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{D}}{\partial t}\right)$$
$$\nabla^2 \mathbf{E} = \boldsymbol{\mu}_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\nabla^2 \mathbf{E} = \frac{\mathbf{1}}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

On the other hand, starting from $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$, we get

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla \times \frac{\partial \mathbf{D}}{\partial t}$$
$$\nabla (\nabla . \mathbf{H}) - \nabla^2 \mathbf{H} = \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E})$$
$$-\nabla^2 \mathbf{H} = \epsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{B}}{\partial t}\right)$$
$$\nabla^2 \mathbf{H} = \epsilon_0 \boldsymbol{\mu}_0 \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{H}}{\partial t}\right)$$
$$\nabla^2 \mathbf{H} = \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

8.7 Commutator Algebra

Definition 35 *Let* $(\mathbb{F}, +, \boldsymbol{.})$ *be a field and* $\hat{A}$,$\hat{B} \in \mathbb{F}$ *The* ***commutator*** *of these two elements is defined by*

$$(\hat{A} \boldsymbol{.} \hat{B}) - (\hat{A} \boldsymbol{.} \hat{B})$$

and the ***anticommutator*** *is defined by*

$$(\hat{A} \boldsymbol{.} \hat{B}) + (\hat{B} \boldsymbol{.} \hat{A})$$

The former is denoted by $[\hat{A}, \hat{B}]$ and the latter is denoted by $\{\hat{A}, \hat{B}\}$. Two elements commute if $[\hat{A}, \hat{B}] = 0$ and anti-commute if $\{\hat{A}, \hat{B}\} = 0$

Proposition 36 $\forall$ $\hat{A}, \hat{B}$,$\hat{C} \in \mathbb{F}$ *and scalar* α

- $[\hat{A}, \hat{A}] = 0$ *(self-commutativity)*
- $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$ *(antisymmetry)*
- $[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$ *(linearity)*
- $[\hat{A}, \hat{B}]^\dagger = [\hat{B}^\dagger, \hat{A}^\dagger]$ *(Hermition conjugate of a commutator)*
- $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} +, \hat{B}[\hat{A}, \hat{C}]$ *(distributivity)*
- $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] +, [\hat{A}, \hat{C}]\hat{B}$ *(distributivity)*
- $[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0$ *(Jacobi identity)*
- $[\hat{A}, \alpha] = 0$ *(commutativity with a scalar)*

The proof for each follows directly from the above property. An interesting consequence of these properties are the uncertainty relations, from which stems Hiesenberg's uncertainty relation

Definition 37 ***Uncertainty*** *in a Hermitian operator* $\hat{A}$ *is defined as*

$$\triangle\hat{A} = \hat{A} - \left\langle \hat{A} \right\rangle$$

. where $\left\langle \hat{A} \right\rangle = \langle\psi|\hat{A}|\psi\rangle$ *denotes the expectation value of* $\hat{A}$ *with respect to a normalised* $|\psi\rangle$

Lemma 38 *If* $\hat{A}$ *is Hermitian, then so is* $\triangle\hat{A}$

Definition 39 *The* ***uncertainty*** $\triangle A$ *of any Hermitian observable* $\hat{A}$ *is defined by* $\triangle A = \sqrt{\left(\triangle\hat{A}\right)^2}$

Lemma 40 $(\triangle A)^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$

Proof. From the definition of $\triangle\hat{A}$, it follows that

$$\begin{aligned}\left(\triangle\hat{A}\right)^2 &= \left(\hat{A}-\left\langle\hat{A}\right\rangle\right)^2 \\ &= \left(\hat{A}-\left\langle\hat{A}\right\rangle\right)\left(\hat{A}-\left\langle\hat{A}\right\rangle\right) \\ &= \hat{A}^2-\left\langle\hat{A}\right\rangle\hat{A}-\hat{A}\left\langle\hat{A}\right\rangle+\left\langle\hat{A}\right\rangle^2\end{aligned}$$

From $\left\langle\hat{A}\right\rangle\hat{A}$, we have $\langle\psi|A|\psi\rangle\hat{A}=\hat{A}\langle\psi|A|\psi\rangle=\hat{A}\left\langle\hat{A}\right\rangle$ Then, $\left(\triangle\hat{A}\right)^2=\hat{A}^2-2\hat{A}\left\langle\hat{A}\right\rangle+\left\langle\hat{A}\right\rangle^2$. Hence

$$\begin{aligned}\left\langle\left(\triangle\hat{A}\right)^2\right\rangle &= \langle\psi|\left(\triangle\hat{A}\right)^2|\psi\rangle=\langle\psi|\left(\hat{A}^2-2\hat{A}\left\langle\hat{A}\right\rangle+\left\langle\hat{A}\right\rangle^2\right)|\psi\rangle \\ &= \langle\psi|\hat{A}^2|\psi\rangle-2\langle\psi|\hat{A}\left\langle\hat{A}\right\rangle|\psi\rangle+\langle\psi|\left\langle\hat{A}\right\rangle^2|\psi\rangle \\ &= \langle\psi|\hat{A}^2|\psi\rangle-2\langle\psi|\hat{A}\left(\langle\psi|A|\psi\rangle\right)|\psi\rangle+\langle\psi|\left(\langle\psi|\hat{A}|\psi\rangle\right)^2|\psi\rangle \\ &= \langle\psi|\hat{A}^2|\psi\rangle-2\left(\langle\psi|A|\psi\rangle\right)\langle\psi|\hat{A}|\psi\rangle+\langle\psi|\hat{A}|\psi\rangle^2\langle\psi|\psi\rangle \\ &= \langle\psi|\hat{A}^2|\psi\rangle-2\left(\langle\psi|A|\psi\rangle\right)^2+\langle\psi|\hat{A}|\psi\rangle^2 \\ &= \langle\psi|\hat{A}^2|\psi\rangle-\langle\psi|\hat{A}|\psi\rangle^2 \\ &= \langle\hat{A}^2\rangle-\langle\hat{A}\rangle^2\end{aligned}$$

■

Theorem 41 *Let $\hat{A},\hat{B}$ be any two Hermitian operators. Then, $\triangle A\triangle B\geq\frac{1}{2}\left|\left\langle[\hat{A},\hat{B}]\right\rangle\right|$*

Proof. Let H be a Hilbert space and $|\psi\rangle\in H$. If $\triangle\hat{A}|\psi\rangle=\left(\hat{A}-\left\langle\hat{A}\right\rangle\right)|\psi\rangle=|\chi\rangle$ and $\triangle\hat{B}|\psi\rangle=\left(\hat{B}-\left\langle\hat{B}\right\rangle\right)|\psi\rangle=|\phi\rangle$, then

$\langle\psi|\left(\triangle\hat{A}\right)^2|\psi\rangle\langle\psi|\left(\triangle\hat{B}\right)^2|\psi\rangle=\langle\left(\triangle\hat{A}\right)^2\rangle\langle\left(\triangle\hat{B}\right)^2\rangle$
$=\langle\chi|\chi\rangle\langle\phi|\phi\rangle$
$\geq|\langle\chi|\phi\rangle|^2$
$=\left|\langle\psi|\triangle\hat{A}\triangle\hat{B}|\psi\rangle\right|^2=\left|\langle\triangle\hat{A}\triangle\hat{B}\rangle\right|^2$
Also, it can be inferred that $\left\langle\triangle\hat{A}\triangle\hat{B}\right\rangle=\frac{1}{2}\left\langle\left[\hat{A},\hat{B}\right]\right\rangle+\frac{1}{2}\left\langle\left\{\hat{A},\hat{B}\right\}\right\rangle$
or $\left|\left\langle\triangle\hat{A}\triangle\hat{B}\right\rangle\right|^2=\frac{1}{4}\left|\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right|^2+\frac{1}{4}\left|\left\langle\left\{\hat{A},\hat{B}\right\}\right\rangle\right|^2$
$\Rightarrow\left|\left\langle\triangle\hat{A}\triangle\hat{B}\right\rangle\right|^2\geq\frac{1}{4}\left|\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right|^2$
Since $\langle\left(\triangle\hat{A}\right)^2\rangle\langle\left(\triangle\hat{B}\right)^2\rangle\geq\left|\langle\triangle\hat{A}\triangle\hat{B}\rangle\right|^2$ and $\left|\left\langle\triangle\hat{A}\triangle\hat{B}\right\rangle\right|^2\geq\frac{1}{4}\left|\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right|^2$
we have $\langle\triangle\hat{A}\rangle\langle\triangle\hat{B}\rangle\geq\frac{1}{4}\left|\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right|$ ■
This is the uncertainty relation

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